Spectral decimation of the magnetic Laplacian on the Sierpinski gasket: Hofstadter's butterfly, determinants, and loop soup entropy

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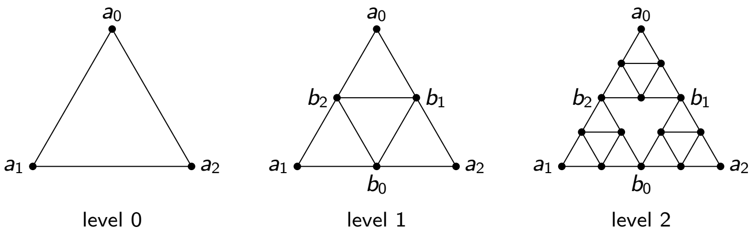
The Sierpinski gasket (SG)

We denote SG on level N by $G_N = (V_N, E_N)$ where V_N is the vertex set, and E_N is the edge set.



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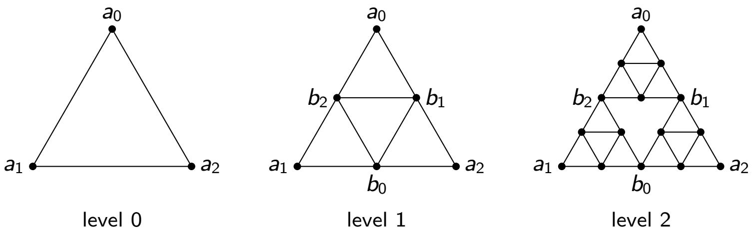


Remark

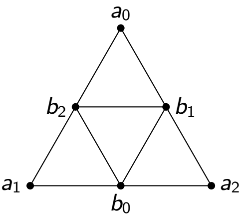
1. Let F_i be the contraction mappings for $i = 0, 1, 2$. Then the infinite SG is the unique nonempty compact set K such that

$$K = \bigcup_{i=0}^{2} F_i(K)$$

2. $\#V_N = \frac{3^{N+1}+3}{2}$
3. SG is **self-similar**

The combinatorial graph Laplacian

Example: $G = (V, E)$, the level 1 SG



$$D_G(i.j) = \begin{cases} \deg(i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad A_G(i,j) = \begin{cases} 1 & \text{if } i \sim j \\ 0 & otherwise \end{cases}$$

$$D = \begin{array}{c|ccc|ccc} & a_0 & a_1 & a_2 & b_0 & b_1 & b_2 \\ \hline a_0 & 2 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 2 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 2 & 0 & 0 & 0 \\ \hline b_0 & 0 & 0 & 0 & 4 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 4 & 0 \\ b_2 & 0 & 0 & 0 & 0 & 0 & 4 \end{array} \qquad A = \begin{array}{c|ccc|ccc} & a_0 & a_1 & a_2 & b_0 & b_1 & b_2 \\ \hline a_0 & 0 & 0 & 0 & 0 & 1 & 1 \\ a_1 & 0 & 0 & 0 & 1 & 0 & 1 \\ a_2 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline b_0 & 0 & 1 & 1 & 0 & 1 & 1 \\ b_1 & 1 & 0 & 1 & 1 & 0 & 1 \\ b_2 & 1 & 1 & 0 & 1 & 1 & 0 \end{array}$$

The **combinatorial graph Laplacian** of level 1 SG is $\Delta_G = D_G - A_G$.

The magnetic Laplacian

We can normalize Δ_G by the degree to obtain the **probabilistic graph Laplacian** $\mathcal{L}_G = D_G^{-1}\Delta_G$, or

$$(\mathcal{L}_G u)(x) = \frac{1}{\deg_G(x)} \sum_{y\sim x}(u(x) - u(y)), \quad u \in \mathbb{R}^V$$

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To obtain the magnetic Laplacian, We assign a set of unit complex values (**complex line bundle**) to replace the 1's in the adjacency matrix such that $\omega_{ij} = \omega_{ji}^{-1}$ for all i, j in V_N. The magnetic Laplacian on the level-N gasket graph G_N endowed with the set of weights ω is defined as

$$(\mathcal{L}_N^{\omega} u)(x) = \sum_{y \sim x} \frac{1}{\deg_{G_N}(x)} (u(x) - \omega_{xy} u(y)), \quad u \in \mathbb{C}^V$$

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Remark

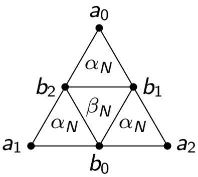
$\mathcal{L}_N^\omega$ is self-adjoint on $L^2(V_N, \deg_{G_N})$, so it has real eigenvalues.

Magnetic fluxes

Definition

The magnetic flux through each smallest upright (resp. downright) triangle on level N equals α_N (resp. β_N).

Suppose that the figure below is part of a level N SG:



$$e^{2\pi i \alpha_N} = \omega_{a_1 b_0}\omega_{b_0 b_2}\omega_{b_2 a_1} = \omega_{b_0 a_2}\omega_{a_2 b_1}\omega_{b_1 b_0}$$

$$e^{2\pi i \beta_N} = \omega_{b_0 b_1}\omega_{b_1 b_2}\omega_{b_2 b_0} \quad e^{-2\pi i \beta_N} = \omega_{b_0 b_2}\omega_{b_2 b_1}\omega_{b_1 b_0}$$

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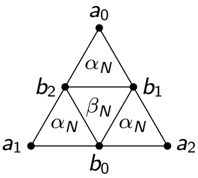

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Remark

Having uniform magnetic field over SG implies $\alpha_N = \beta_N$.

The magnetic spectrum

Question

What is the magnetic spectrum when SG is subject to uniform magnetic field?

[1] Bellissard, 1990; Ghez et. al, 1987

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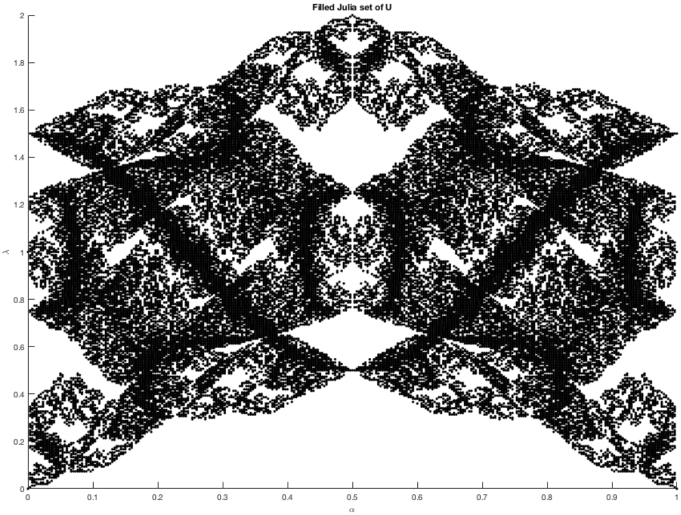
Answer:[1]



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Case I: magnetic spectrum under (half-) integer flux, $\alpha, \beta \in \{0, \frac{1}{2}\}$ (Chen–G. '19)

$$\mathcal{L}_N^{(0,0)} \xrightarrow{R(0,0,\cdot)} \mathcal{L}_{N-1}^{(0,0)} \xrightarrow{R(0,0,\cdot)} \mathcal{L}_{N-2}^{(0,0)} \longrightarrow \cdots \longrightarrow \mathcal{L}_0^{(0,0)}$$

	$\sigma(\mathcal{L}_N^{(\alpha,\beta)})$	Respective multiplicity
$(\alpha,\beta)=(0,0)$[2]	$0, \frac{3}{2}, R(0,0,\cdot))^{-k}(\frac{3}{4}), R(0,0,\cdot))^{-k}(\frac{5}{4})$	$1, \frac{3^N+3}{2}, \frac{3^{N-k-1}+3}{2}, \frac{3^{N-k-1}-1}{2}$
$(\alpha,\beta)=(\frac{1}{2},\frac{1}{2})$	$\frac{1}{2}, \frac{3}{4}, \frac{5}{4}, 2,$ $\left(R\left(\frac{1}{2},\frac{1}{2},\cdot\right)\right)^{-1}(R_1 \cup R_2)$	$\frac{3^N+3}{2}, \frac{3^{N-1}-1}{2}, \frac{3^{N-1}+3}{2}, 1,$ $\frac{3^{N-k-2}+3}{2}, \frac{3^{N-k-2}-1}{2}$
$(\alpha,\beta)=(\frac{1}{2},0)$	$\frac{1}{2}, 1, \frac{5}{4}, \frac{7}{4}, \left(R\left(\frac{1}{2},0,\cdot\right)\right)^{-1}\left(\left\{\frac{3}{4},\frac{5}{4}\right\}\right),$ $\left(R\left(\frac{1}{2},0,\cdot\right)\right)^{-1} \circ \left(R\left(\frac{1}{2},\frac{1}{2},\cdot\right)\right)^{-1}(R_3 \cup R_4)$	$\frac{3^N+3}{2}, 1, \frac{3^{N-1}-1}{2}, \frac{3^{N-1}+3}{2}, \frac{3^{N-2}-1}{2},$ $\frac{3^{N-2}+3}{2}, \frac{3^{N-k-3}+3}{2}, \frac{3^{N-k-3}-1}{2}$
$(\alpha,\beta)=(0,\frac{1}{2})$	$\{\frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}\}, \left(R\left(0,\frac{1}{2},\cdot\right)\right)^{-1}\left(\left\{\frac{3}{4},\frac{5}{4}\right\}\right)$ $\left(R\left(0,\frac{1}{2},\cdot\right)\right)^{-1} \circ \left(R\left(\frac{1}{2},\frac{1}{2},\cdot\right)\right)^{-1}(R_3 \cup R_4)$	$\frac{3^{N-1}+3}{2}, \frac{3^{N-1}-1}{2}, 1, \frac{3^N+3}{2}, \frac{3^{N-2}-1}{2},$ $\frac{3^{N-2}+3}{2}, \frac{3^{N-k-3}+3}{2}, \frac{3^{N-k-3}-1}{2}$

where $R_1 = \bigcup_{k=0}^{N-2} (R(0,0,\cdot))^{-k}\left(\frac{3}{4}\right) \quad R_2 = \bigcup_{k=0}^{N-3} (R(0,0,\cdot))^{-k}\left(\frac{5}{4}\right)$

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[2] $R(\alpha,\beta,\lambda)$ is the decimation function, $k = \{0, 1, \cdots N-1\}$, Fukushima & Shima, 1992

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\begin{array}{ccccccccc}
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& \nearrow{\scriptstyle R(\frac{1}{2},\frac{1}{2},\cdot)} & & \nearrow{\scriptstyle R(\frac{1}{2},\frac{1}{2},\cdot)} & & & & & \\
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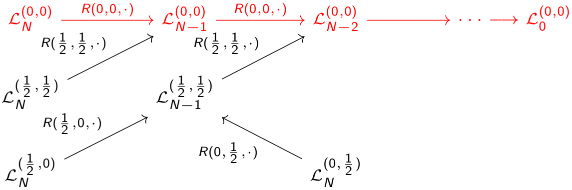

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Other cases

Theorem: Magnetic spectra under non-(half-)integer fluxes (Chen–G. '19)

Let $\mathcal{E}(\alpha_N, \beta_N)$ be the **exceptional set for spectral decimation**. Suppose not both of α_N and β_N are in $\{0, \frac{1}{2}\}$. Then

$$\sigma\left(\mathcal{L}_N^{(\alpha_N,\beta_N)}\right) = \left\{\lambda \in \mathbb{R} \setminus \mathcal{E}(\alpha_N, \beta_N) : R(\alpha_N, \beta_N, \lambda) \in \sigma\left(\mathcal{L}_{N-1}^{(\alpha_{N-1},\beta_{N-1})}\right)\right\}$$
$$\sqcup \left\{\lambda : \mathcal{D}(\beta_N, \lambda) = 0,\ \text{mult}\left(\mathcal{L}_N^{(\alpha_N,\beta_N)}, \lambda\right) > 0\right\} \sqcup \left\{\begin{array}{ll} \frac{3}{2}, & \text{if } \alpha_N = 0 \\ \frac{1}{2}, & \text{if } \alpha_N = \frac{1}{2} \end{array}\right\},$$

Spectral decimation

Spectral decimation is a process in which we project the eigenspace of $\mathcal{L}_N^\omega$ to that of $\mathcal{L}_{N-1}^\Omega$. We do so by computing the Schur complement

Schur complement

Define the **Schur complement** of $\mathcal{L}_N^\omega - \lambda I$ with respect to the minor $D - \lambda I$ as

$$S_N^\omega(\lambda) := (A - \lambda I) - B(D - \lambda I)^{-1} C,$$

where

$$\begin{aligned} A &: \ell(V_{N-1}) \to \ell(V_{N-1}), \\ B &: \ell(V_N \setminus V_{N-1}) \to \ell(V_{N-1}), \\ C &: \ell(V_{N-1}) \to \ell(V_N \setminus V_{N-1}), \\ D &: \ell(V_N \setminus V_{N-1}) \to \ell(V_N \setminus V_{N-1}), \\ S_N^\omega(\lambda) &: \ell(V_{N-1}) \to \ell(V_{N-1}) \end{aligned}$$

and make the connection by writing

$$S_N^\omega(\alpha, \beta, \lambda) = \phi(\alpha, \beta, \lambda)(\mathcal{L}_{N-1}^\Omega - R(\alpha, \beta, \lambda)). \qquad \lambda \in \mathbb{C},$$

Then, $\mathcal{L}_N^\omega$ and $\mathcal{L}_{N-1}^\Omega$ are said to be spectrally similar, and if $\lambda \notin \mathcal{E}(\alpha_N, \beta_N)$, then

$$\lambda \in \sigma(\mathcal{L}_N^\omega) \Leftrightarrow R(\alpha_N, \beta_N, \lambda) \in \sigma(\mathcal{L}_{N-1}^\Omega)$$

Recall that we write

$$S_N^{\omega}(\alpha,\beta,\lambda)=\phi(\alpha,\beta,\lambda)(\mathcal{L}_{N-1}^{\Omega}-R(\alpha,\beta,\lambda))$$

and if $\lambda\notin\mathcal{E}(\alpha_N,\beta_N)$, then

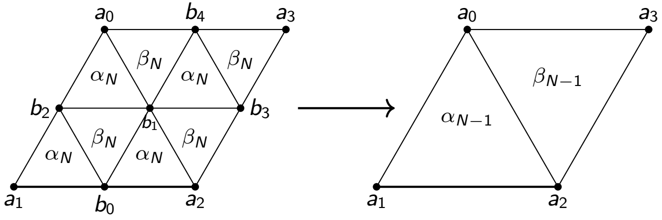
$$\lambda\in\sigma(\mathcal{L}_N^{\omega})\Leftrightarrow R(\lambda)\in\sigma(\mathcal{L}_{N-1}^{\Omega})$$

Computations

$$\begin{aligned}
R(\alpha,\beta,\lambda)&=1+\frac{A(\alpha,\beta,\lambda)-64\mathcal{D}(\beta,\lambda)(1-\lambda)}{16|\Psi(\alpha,\beta,\lambda)|},\\
\phi(\alpha,\beta,\lambda)&=\frac{|\Psi(\alpha,\beta,\lambda)|}{4\mathcal{D}(\beta,\lambda)},\\
A(\alpha,\beta,\lambda)&=16\lambda^2-(32+4\cos(2\pi\alpha))\lambda+15+4\cos(2\pi\alpha)+\cos(2\pi(\alpha+\beta)),\\
\mathcal{D}(\beta,\lambda)&=-\lambda^3+3\lambda^2-\frac{45}{16}\lambda+\frac{13}{16}-\frac{1}{32}\cos(2\pi\beta),\\
\Psi(\alpha,\beta,\lambda)&=(1-\lambda)^2-\frac{1}{16}+\frac{1-\lambda}{4}(2e^{-2\pi i\alpha}+e^{-2\pi i(2\alpha+\beta)})\\
&\quad+\frac{1}{16}(e^{-4\pi i\alpha}+2e^{-2\pi i(\alpha+\beta)}),\\
\mathcal{E}(\alpha,\beta)&=\{\lambda\in\mathbb{R}:\Psi(\alpha,\beta,\lambda)=0\text{ or }\mathcal{D}(\beta,\lambda)=0\}
\end{aligned}$$

Flux changes in spectral decimation

$$\Omega_{a_1 a_2}(\alpha,\beta,\lambda) = \omega_{a_1 b_0}\omega_{b_0 a_2} e^{2\pi i \theta(\alpha,\beta,\lambda)}$$



Therefore,

$$\theta(\alpha,\beta,\lambda) = \frac{\arg\ \Psi(\alpha,\beta,\lambda)}{2\pi} \qquad (\arg : \mathbb{C} \to [0,2\pi)),$$

$$\alpha_{N-1} = \alpha_{\downarrow}(\alpha_N,\beta_N,\lambda) \quad \text{and} \quad \beta_{N-1} = \beta_{\downarrow}(\alpha_N,\beta_N,\lambda),$$

$$\alpha_{\downarrow}(\alpha,\beta,\lambda) = 3\alpha + \beta - 3\theta(\alpha,\beta,\lambda) \pmod 1,$$

$$\beta_{\downarrow}(\alpha,\beta,\lambda) = 3\beta + \alpha + 3\theta(\alpha,\beta,\lambda) \pmod 1$$

3-parameter non-rational function

$$\mathcal{U}(\alpha,\beta,\lambda) = (3\alpha+\beta-3\theta, 3\beta+\alpha+3\theta, R(\alpha,\beta,\lambda))$$

Other cases

Theorem: Magnetic spectra under non-(half-)integer fluxes (Chen–G. '19)

Let $\mathcal{E}(\alpha_N, \beta_N)$ be the **exceptional set for spectral decimation**. Suppose not both of α_N and β_N are in $\{0, \frac{1}{2}\}$. Then

$$\sigma\left(\mathcal{L}_N^{(\alpha_N,\beta_N)}\right) = \left\{\lambda \in \mathbb{R} \setminus \mathcal{E}(\alpha_N, \beta_N) : R(\alpha_N, \beta_N, \lambda) \in \sigma\left(\mathcal{L}_{N-1}^{(\alpha_{N-1},\beta_{N-1})}\right)\right\}$$
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The exceptional set for spectral decimation

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Is the dynamical spectrum equal to the actual spectrum of the original operator? This is a question with no answer yet.

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Recall that we write

$$S_N^{\omega}(\alpha,\beta,\lambda) = \phi(\alpha,\beta,\lambda)(\mathcal{L}_{N-1}^{\Omega} - R(\alpha,\beta,\lambda)), \quad \phi(\alpha,\beta,\lambda) = \frac{|\Psi(\alpha,\beta,\lambda)|}{4\mathcal{D}(\beta,\lambda)},$$

$$S_N^{\omega}(\alpha,\beta,\lambda) = (A-\lambda I) - B(D-\lambda I)^{-1}C,$$

so naturally,

$$\mathcal{E}(\alpha,\beta) = \{\lambda \in \mathbb{R} : \Psi(\alpha,\beta,\lambda) = 0 \text{ or } \mathcal{D}(\beta,\lambda) = 0\}$$

Given any fluxes α and β, the **exceptional set** (for spectral decimation of $\mathcal{L}_N^{\omega}$) $\mathcal{E}(\alpha,\beta)$ consists of:

- The three zeros of $\mathcal{D}(\beta,\cdot)$; and
- The corresponding values x in the table below if any of the conditions in the first column is met.

Condition	Value x to be added to $\mathcal{E}(\alpha,\beta)$
$\alpha = 0$	$\frac{3}{2}$
$\alpha = \frac{1}{2}$	$\frac{1}{2}$
$3\alpha + \beta = \frac{1}{2} \pmod 1$	$1 + \frac{1}{2}\cos(2\pi\alpha)$

where $\mathcal{D}(\beta,\lambda) = -\lambda^3 + 3\lambda^2 - \frac{45}{16}\lambda + \frac{13}{16} - \frac{1}{32}\cos(2\pi\beta)$.

Additional analysis on the exceptional set

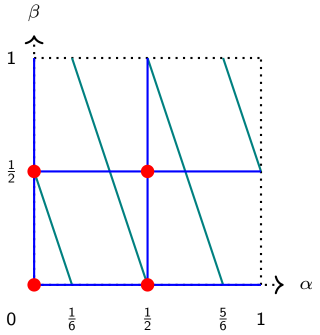
$$\mathcal{E}(\alpha, \beta) = \{\lambda \in \mathbb{R} : \Psi(\alpha, \beta, \lambda) = 0 \text{ or } \mathcal{D}(\beta, \lambda) = 0\}$$

<u>Case I</u>: $\alpha, \beta \in \{0, \frac{1}{2}\}$. Spectral decimation can be carried out explicitly.

<u>Case II</u>: Only one of α and β is in $\{0, \frac{1}{2}\}$. There is only one $\mathbb{R}$-valued zero of $\Psi(\alpha, \beta, \cdot)$.

<u>Case III</u>: $3\alpha + \beta = \frac{1}{2} \pmod 1$, excluding flux values already discussed in Cases I & II. There is only one $\mathbb{R}$-valued zero of $\Psi(\alpha, \beta, \cdot)$.

<u>Case IV</u>: The remaining case. There are no $\mathbb{R}$-valued zeros of $\Psi(\alpha, \beta, \cdot)$.



There is a standard way to analyze the exceptional set using complex analysis.[3] However, it is necessary to use real analysis in our case.

[3] Bajorin et. al, 2008 -

Other cases

$$\mathcal{E}(\alpha,\beta)=\{\lambda\in\mathbb{R}:\Psi(\alpha,\beta,\lambda)=0 \text{ or } \mathcal{D}(\beta,\lambda)=0\}$$

Case I: $\alpha,\beta\in\{0,\frac{1}{2}\}$. Spectral decimation can be carried out explicitly.

Case II: Only one of α and β is in $\{0,\frac{1}{2}\}$. There is only one $\mathbb{R}$-valued zero of $\Psi(\alpha,\beta,\cdot)$.

Case III: $3\alpha+\beta=\frac{1}{2}$ (mod 1), excluding flux values already discussed in Cases I & II. There is only one $\mathbb{R}$-valued zero of $\Psi(\alpha,\beta,\cdot)$.

Case IV: The remaining case. There are no $\mathbb{R}$-valued zeros of $\Psi(\alpha,\beta,\cdot)$.

Theorem: Magnetic spectra under non-(half-)integer fluxes (Chen–G. '19)

Let $\mathcal{E}(\alpha_N,\beta_N)$ be the **exceptional set for spectral decimation**. Suppose not both of α_N and β_N are in $\{0,\frac{1}{2}\}$. Then

$$\sigma\left(\mathcal{L}_N^{(\alpha_N,\beta_N)}\right)=\left\{\lambda\in\mathbb{R}\setminus\mathcal{E}(\alpha_N,\beta_N):R(\alpha_N,\beta_N,\lambda)\in\sigma\left(\mathcal{L}_{N-1}^{(\alpha_{N-1},\beta_{N-1})}\right)\right\}$$

$$\sqcup\left\{\lambda:\mathcal{D}(\beta_N,\lambda)=0,\ \text{mult}\left(\mathcal{L}_N^{(\alpha_N,\beta_N)},\lambda\right)>0\right\}\sqcup\left\{\begin{array}{ll}\frac{3}{2}, & \text{if } \alpha_N=0\\ \frac{1}{2}, & \text{if } \alpha_N=\frac{1}{2}\end{array}\right\},$$

Magnetic spectra

Theorem: Magnetic spectra under non-(half-)integer fluxes (Chen–G. '19)

Let $\mathcal{E}(\alpha_N, \beta_N)$ be the **exceptional set for spectral decimation**. Suppose not both of α_N and β_N are in $\{0, \frac{1}{2}\}$. Then

$$\sigma\left(\mathcal{L}_N^{(\alpha_N,\beta_N)}\right) = \left\{\lambda \in \mathbb{R} \setminus \mathcal{E}(\alpha_N, \beta_N) : R(\alpha_N, \beta_N, \lambda) \in \sigma\left(\mathcal{L}_{N-1}^{(\alpha_{N-1},\beta_{N-1})}\right)\right\}$$

$$\sqcup \left\{\lambda : \mathcal{D}(\beta_N, \lambda) = 0,\ \text{mult}\left(\mathcal{L}_N^{(\alpha_N,\beta_N)}, \lambda\right) > 0\right\} \sqcup \left\{ \begin{array}{ll} \frac{3}{2}, & \text{if } \alpha_N = 0 \\ \frac{1}{2}, & \text{if } \alpha_N = \frac{1}{2} \end{array} \right\},$$

Theorem: Magnetic spectra under (half-) integer fluxes (Chen–G. '19)

(α, β)	$\sigma(\mathcal{L}_N^{(\alpha,\beta)})$	Respective multiplicity
$(0,0)^*$	$0,\ \frac{3}{2}, R(0,0,\cdot))^{-k}(\frac{3}{4}), R(0,0,\cdot))^{-k}(\frac{5}{4})$	$1,\ \frac{3^N+3}{2},\ \frac{3^{N-k-1}+3}{2},\ \frac{3^{N-k-1}-1}{2}$
$(\frac{1}{2}, \frac{1}{2})$	$\frac{1}{2}, \frac{3}{4}, \frac{5}{4}, 2,$ $\left(R\left(\frac{1}{2}, \frac{1}{2}, \cdot\right)\right)^{-1}(R_1 \cup R_2)$	$\frac{3^N+3}{2},\ \frac{3^{N-1}-1}{2},\ \frac{3^{N-1}+3}{2},\ 1,$ $\frac{3^{N-k-2}+3}{2},\ \frac{3^{N-k-2}-1}{2}$
$(\frac{1}{2}, 0)$	$\frac{1}{2}, 1, \frac{5}{4}, \frac{7}{4}, \left(R\left(\frac{1}{2}, 0, \cdot\right)\right)^{-1}\left(\left\{\frac{3}{4}, \frac{5}{4}\right\}\right),$ $\left(R\left(\frac{1}{2}, 0, \cdot\right)\right)^{-1} \circ \left(R\left(\frac{1}{2}, \frac{1}{2}, \cdot\right)\right)^{-1}(R_3 \cup R_4)$	$\frac{3^N+3}{2},\ 1,\ \frac{3^{N-1}-1}{2},\ \frac{3^{N-1}+3}{2},\ \frac{3^{N-2}-1}{2},$ $\frac{3^{N-2}+3}{2},\ \frac{3^{N-k-3}+3}{2},\ \frac{3^{N-k-3}-1}{2}$
$(0, \frac{1}{2})$	$\left\{\frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}\right\}, \left(R\left(0, \frac{1}{2}, \cdot\right)\right)^{-1}\left(\left\{\frac{3}{4}, \frac{5}{4}\right\}\right)$ $\left(R\left(0, \frac{1}{2}, \cdot\right)\right)^{-1} \circ \left(R\left(\frac{1}{2}, \frac{1}{2}, \cdot\right)\right)^{-1}(R_3 \cup R_4)$	$\frac{3^{N-1}+3}{2},\ \frac{3^{N-1}-1}{2},\ 1,\ \frac{3^N+3}{2},\ \frac{3^{N-2}-1}{2},$ $\frac{3^{N-2}+3}{2},\ \frac{3^{N-k-3}+3}{2},\ \frac{3^{N-k-3}-1}{2}$

Determinants of the magnetic Laplacian under (half-) integer fluxes (Chen–G. '19)

$$\det(\mathcal{L}_N^{(\frac{1}{2},\frac{1}{2})}) = \frac{1}{\kappa(G_N)} \cdot 2^{\frac{3^N}{2}+\frac{3}{2}} \cdot 3^{\frac{3^{N-1}}{2}-N-\frac{3}{2}} \cdot 5^{\frac{3^{N-1}}{2}+\frac{3}{2}}$$
$$\times \left[\prod_{k=0}^{N-2}\left(H(k)+\frac{1}{2}\right)^{\frac{3^{N-k-2}+3}{2}}\right]\left[\prod_{k=0}^{N-3}\left(H(k)+\frac{5}{2}\right)^{\frac{3^{N-k-2}-1}{2}}\right],$$

where $H(0) = 26.5$, and for $k \geq 1$, $H(k) = [H(k-1)]^2 - \frac{15}{4}$.

$$\det(\mathcal{L}_N^{(\frac{1}{2},0)}) = \frac{1}{\kappa(G_N)} \cdot 2^{\frac{13}{6}3^{N-1}-\frac{5}{2}} \cdot 3^{\frac{3^{N-2}}{2}-N-\frac{3}{2}} \cdot 5^{\frac{5}{2}3^{N-2}-1} \cdot 7^{\frac{3^{N-1}}{2}+\frac{3}{2}} \cdot 17^{\frac{3^{N-2}}{2}+\frac{3}{2}}$$
$$\times \left[\prod_{k=0}^{N-3}\left(\tilde{H}(k)+\frac{1}{2}\right)^{\frac{3^{N-k-3}+3}{2}}\right]\left[\prod_{k=0}^{N-4}\left(\tilde{H}(k)+\frac{5}{2}\right)^{\frac{3^{N-k-3}-1}{2}}\right],$$

where $\tilde{H}(0) = 302.5$, and for $k \geq 1$, $\tilde{H}(k) = [\tilde{H}(k-1)]^2 - \frac{15}{4}$.

$$\det(\mathcal{L}_N^{(0,\frac{1}{2})}) = \frac{1}{\kappa(G_N)} \cdot 2^{\frac{13}{6}3^{N-1}-\frac{5}{2}} \cdot 3^{\frac{7}{3}3^{N-1}-N+3} \cdot 7^{\frac{3^{N-2}}{2}-\frac{1}{2}}$$
$$\times \left[\prod_{k=0}^{N-3}\left(\hat{H}(k)+\frac{1}{2}\right)^{\frac{3^{N-k-3}+3}{2}}\right]\left[\prod_{k=0}^{N-4}\left(\hat{H}(k)+\frac{5}{2}\right)^{\frac{3^{N-k-3}-1}{2}}\right],$$

where $\hat{H}(0) = 86.5$, and for $k \geq 1$, $\hat{H}(k) = [\hat{H}(k-1)]^2 - \frac{15}{4}$.

A **cycle-rooted spanning forest (CRSF)** is a spanning forest whose connected components are unicycles (a tree plus an edge to form a single cycle).

Matrix-CRSF Theorem[4]: Let $\mathcal{L}^{\omega}_{(G,\mathbf{c})}$ be the line bundle Laplacian, then

$$\det\left(\mathcal{L}^{\omega}_{(G,\mathbf{c})}\right) = \sum_{\text{OCRSFs}} \prod_{e \in \text{bushes}} \mathbf{c}(e) \prod_{\gamma \in \text{cycles}} \mathbf{C}(\gamma)\,(1-\omega(\gamma))\,.$$

Asymptotic complexity (tree entropy[5]):

$$\mathfrak{h}(G_\infty, \mathcal{L}^{\omega}_{\infty}) := \lim_{N\to\infty} \frac{\log\left(\kappa(G_N)\det(\mathcal{L}^{\omega}_{N})\right)}{|V_N|}$$

Loop soup entropy:

$$\mathfrak{h}_{\text{loop}}(G_\infty, \mathcal{L}^{\omega}_{\infty}) := \mathfrak{h}(G_\infty, \mathcal{L}^{\omega}_{\infty}) - \mathfrak{h}(G_\infty, \mathcal{L}^{\text{Id}}_{\infty}).$$

Probabilistic interpretation:

$$\lim_{N\to\infty} \lim_{c\downarrow 0} \frac{1}{|V_N|} \log \mathbb{P}^{(\alpha,\beta)}_{N,c}\,[\text{no loops}] = -\mathfrak{h}_{\text{loop}}(SG, \mathcal{L}^{(\alpha,\beta)}_{\infty})$$

[4] Kenyon, 2011

[5] Lyons, 2005

Thank you for your attention!