Spectral decimation on self-similar fractals: from singularly continuous spectrum to the Hofstadter butterfly

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Motivation: Analysis on nonsmooth domains
Some fractals are nicer than others

Each of these fractals is obtained from a nested sequence of graphs which has *nice, symmetric* replacement rules.
Spectral decimation (= spectral similarity)

Rammal-Toulouse ‘84, Bellissard ‘88, Fukushima-Shima ‘92, Shima ‘96, etc.

A recursive algorithm for identifying the Laplacian spectrum on highly symmetric, finitely ramified self-similar fractals.
Definition (Malozemov-Teplyaev ’03)

Let $H$ and $H_0$ be Hilbert spaces. We say that an operator $H$ on $H$ is spectrally similar to $H_0$ on $H_0$ with functions $\varphi_0$ and $\varphi_1$ if there exists a partial isometry $U : H_0 \rightarrow H$ (that is, $UU^* = I$) such that

$$U(H - z)^{-1}U^* = (\varphi_0(z)H_0 - \varphi_1(z))^{-1} =: \frac{1}{\varphi_0(z)} (H_0 - R(z))^{-1}$$

for any $z \in \mathbb{C}$ for which the two sides make sense.

A common class of examples: $H_0$ subspace of $H$, $U^*$ is an ortho. projection from $H$ to $H_0$. Write $H - z$ in block matrix form w.r.t. $H_0 \oplus H_0^\perp$:

$$H - z = \begin{pmatrix} l_0 - z & \bar{X} \\ X & Q - z \end{pmatrix}.$$ 

Then $U(H - z)^{-1}U^*$ is the inverse of the Schur complement $S(z)$ w.r.t. to the lower-right block of $H - Z$: $S(z) = (l_0 - z) - \bar{X}(Q - z)^{-1}X$.

Issue: There may exist a set of $z$ for which either $Q - z$ is not invertible, or $\varphi_0(z) = 0$. 
Spectral decimation: the main theorem

Spectrum $\sigma(\Delta) = \{ z \in \mathbb{C} : \Delta - z \text{ does not have a bounded inverse} \}$.

**Definition**

The exceptional set for spectral decimation is

$$\mathcal{E}(H, H_0) \overset{\text{def}}{=} \{ z \in \mathbb{C} : z \in \sigma(Q) \text{ or } \varphi_0(z) = 0 \}.$$  

**Theorem (Malozemov-Teplyaev '03)**

Suppose $H$ is spectrally similar to $H_0$. Then for any $z \notin \mathcal{E}(H, H_0)$:

- $R(z) \in \sigma(H_0) \iff z \in \sigma(H)$.
- $R(z)$ is an eigenvalue of $H_0$ iff $z$ is an eigenvalue of $H$. Moreover there is a one-to-one map between the two eigenspaces.

**Consequence:** For an operator $H$ on a self-similar Hilbert space $\mathcal{H}$,

$$\mathcal{J}(R) \subset \sigma(H) \subset \mathcal{J}(R) \cup D,$$

where

- $\mathcal{J}(R)$ is the Julia set of $R$ (= the complement in $\mathbb{C} \cup \{ \infty \}$ of the domain in which $\{ R^n \}_n$ converges uniformly on compact subsets).
- $D$ derives from the exceptional set $\mathcal{E}$.  

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Let \( \Delta \) be the graph Laplacian on \( \mathbb{Z}_+ \) (with Neumann boundary condition at 0), realized as the limit of graph Laplacians on \([0, 2^n] \cap \mathbb{Z}_+\).

If \( z \neq 2 \) and \( R(z) = z(4 - z) \), then

- \( R(z) \in \sigma(-\Delta) \iff z \in \sigma(-\Delta) \).
- \( \sigma(-\Delta) = \mathcal{I}(R) \).
- \( \mathcal{I}(R) \) is the full interval \([0, 4] \).
Generalizing the interval: The $pq$-model

A one-parameter model of 1D fractals parametrized by $p \in (0,1)$. Set $q = 1 - p$.

A triadic interval construction, “next easiest” fractal beyond the dyadic interval.

\[(\Delta_p f)(x) = \sum_y p(x, y)f(y) - f(x)\]

where $p(x, y) \in \{1, p, q\}$ depends on the arrow given below.

Assign probability weights to the three segments:

\[m_1 = m_3 = \frac{q}{1 + q}, \quad m_2 = \frac{p}{1 + q}\]

Then iterate. Let $\pi$ be the resulting self-similar probability measure.
The spectral decimation polynomial is $R(z) = \frac{z(z^2 - 3z + (2pq))}{pq}$.

$$\sigma(-\Delta_n) = \{0, 2\} \cup \bigcup_{m=0}^{n-1} R^{-m}\{1 \pm q\}$$
The spectral decimation polynomial is \( R(z) = \frac{z(z^2-3z+(2+pq))}{pq} \).

\[ \sigma(-\Delta_n) = \{0, 2\} \cup \bigcup_{m=0}^{n-1} R^{-m}\{1 \pm q\} \]
The $pq$-model on $\mathbb{Z}_+$

- $\Delta_p$ is not self-adjoint w.r.t. $\ell^2(\mathbb{Z}_+)$, but is self-adjoint w.r.t. the discretization of the aforementioned self-similar measure $\pi$.
- Let $\Delta_p^+ = D^* \Delta_p D$, where

$$D : \ell^2(\mathbb{Z}_+) \to \ell^2(3\mathbb{Z}_+), \quad (Df)(x) = f(3x).$$

Then $\Delta_p$ is spectrally similar to $\Delta_p^+$. Moreover, $\Delta_p$ and $\Delta_p^+$ are isometrically equivalent (in $L^2(\mathbb{Z}_+)$ or in $L^2(\mathbb{Z}_+, \pi)$).
The \( pq \)-model on \( \mathbb{Z}_+ \)

\[
\begin{array}{cccccccccccccccccccc}
1 & q & p & q & p & q & p & q & p & q & p & q & p & q & p & q & p & q & p & q & p & q & p \end{array}
\]

Spectrum \( \sigma(H) = \{ z \in \mathbb{C} : H - z \text{ does not have a bounded inverse} \} \).

Facts from functional analysis:

- \( \sigma(H) \) is a nonempty compact subset of \( \mathbb{C} \).
- \( \sigma(H) \) equals the disjoint union \( \sigma_{pp}(H) \cup \sigma_{ac}(H) \cup \sigma_{sc}(H) \).
  - pure point spectrum \( \cup \) absolutely continuous spectrum \( \cup \) singularly continuous spectrum


If \( p \neq \frac{1}{2} \), the Laplacian \( \Delta_p \), regarded as an operator on either \( \ell^2(\mathbb{Z}_+) \) or \( L^2(\mathbb{Z}_+, \pi) \), has purely singularly continuous spectrum. The spectrum is the Julia set of the polynomial

\[
R(z) = \frac{z(z^2 - 3z + (2 + pq))}{pq},
\]

which is a topological Cantor set of Lebesgue measure zero.

- One of the simplest realizations of purely singularly continuous spectrum. The mechanism appears to be simpler than those of quasi-periodic or aperiodic Schrodinger operators. (cf. Simon, Jitomirskaya, Avila, Damanik, Gorodetski, etc.)
- See also Grigorchuk-Lenz-Nagnibeda ‘14, ‘16 on spectra of Schreier graphs.
Proof of purely singularly continuous spectrum (when $p \neq \frac{1}{2}$)

Spectral decimation: $\Delta_p$ is spectrally similar to $\Delta_{p^+}$, and they are isometrically equivalent. After taking into account the exceptional set, $R(z) \in \sigma(\Delta_p) \iff z \in \sigma(\Delta_p)$. Notably, the repelling fixed points of $R$, $\{0, 1, 2\}$, lie in $\sigma(\Delta_p)$.

By 1, $\bigcup_{n=0}^{\infty} R^{\circ-n}(0) \subset \sigma(\Delta_p)$. Meanwhile, since $0 \in \mathcal{J}(R)$, $\bigcup_{n=0}^{\infty} R^{\circ-n}(0) = \mathcal{J}(R)$. So $\mathcal{J}(R) \subset \sigma(\Delta_p)$.

If $z \in \sigma(\Delta_p)$, then by 1, $R^{\circ n}(z) \in \sigma(\Delta_p)$ for each $n \in \mathbb{N}$. On the one hand, $\sigma(\Delta_p)$ is compact. On the other hand, the only attracting fixed point of $R$ is $\infty$, so the Fatou set $(\mathcal{J}(R))^c$ contains the basin of attraction of $\infty$, whence non-compact. Infer that $z \notin (\mathcal{J}(R))^c$. So $\sigma(\Delta_p) \subset \mathcal{J}(R)$.
Thus $\sigma(\Delta_p) = \mathcal{J}(R)$. When $p \neq \frac{1}{2}$, $\mathcal{J}(R)$ is a disconnected Cantor set. So $\sigma_{ac}(\Delta_p) = \emptyset$.

Find the formal eigenfunctions corresponding to the fixed points of $R$, and show that none of them are in $\ell^2(\mathbb{Z}^+)$ or in $L^2(\mathbb{Z}^+, \pi)$. Thus none of the fixed points lie in $\sigma_{pp}(\Delta_p)$. By spectral decimation, none of the pre-iterates of the fixed points under $R$ are in $\sigma_{pp}(\Delta_p)$. So $\sigma_{pp}(\Delta_p) = \emptyset$.

Conclude that $\sigma(\Delta_p) = \sigma_{sc}(\Delta_p)$. 
The Sierpinski gasket lattice (SGL)

Let $\Delta$ be the graph Laplacian on $SGL$. If $z \notin \{2, 5, 6\}$ and $R(z) = z(5 - z)$, then

- $R(z) \in \sigma(-\Delta) \iff z \in \sigma(-\Delta)$.
- $\sigma(-\Delta) = \mathcal{J}_R \cup \mathcal{D}$, where $\mathcal{J}_R$ is the Julia set of $R(z)$ and 
  $\mathcal{D} := \{6\} \cup \left( \bigcup_{m=0}^{\infty} R^{-m}\{3\} \right)$.
- $\mathcal{J}_R$ is a disconnected Cantor set.

Thm. (Teplyaev ’98, Quint ’09)
On $SGL$, $\sigma(\Delta) = \sigma_{pp}(\Delta)$.
Eigenfunctions with finite support are complete.

→ Localization due to geometry.
Localized eigenfunctions on $\text{SGL}$

For $z=6$ and $z=3$,

For $z=6$ and $z=5$,
Spectral decimation meets Kirchhoff’s matrix-tree theorem:

On a finite connected graph $G$ on $n$ vertices,

$$\#\{\text{spanning trees on } G\} = \det(-\Delta_G[j]) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i,$$

where $\Delta_G[j]$ is the minor of $\Delta_G$ with the $j$th row and the $j$th column removed, and $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ are the eigenvalues of $-\Delta_G$.

- Counting spanning trees on fractals: Chang–Chen ’05, Teufl–Wagner ’06, Anema–Tsougkas ’16 (uses spectral decimation).
Wave propagation on fractals

\[ u_{tt} = \Delta u, \quad \text{solution is} \quad u(t, x) = \sum_{j=1}^{\infty} c_j \cos(t \sqrt{\lambda_j})e_j(x) \quad \text{where} \quad -\Delta e_j = \lambda_j e_j \]

1. For the \( pq \)-model applied to a compact subinterval of \( \mathbb{R} \), we obtain a good space-time approximation of the solution to the wave equation in Andrews–Bonik–C.–Martin–Teplyaev '17. [Animation]

2. Gives a concrete example of “infinite speed of wave propagation” on fractals.

Anderson localization of \( H = -\Delta + V_\omega \) on fractal lattices?

1. Molchanov: On finitely ramified lattices, \( \sigma_{ac}(H) = \emptyset \) (by the Simon-Wolff method).

2. (Still many unanswered questions ... )
Let $\theta$ be a 1-form on the edge set of a connected graph: $\theta(x, y) = -\theta(y, x)$ for all $x \sim y$. The line bundle Laplacian on the vertex set is given by

$$(\Delta_\theta f)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} \left( f(x) - e^{i\theta(x, y)} f(y) \right), \quad f : V \to \mathbb{R}.$$ 

Line bundle = Vector bundle with a $U(1)$ connection = $\{ e^{i\theta(x, y)} \}_{(x, y) \in E}$.

We are interested in a choice of $\theta$ which corresponds to “constant magnetic field” through the SG lattice:
Flux around a cycle, $\sum_{e \in \text{cycle}} \theta(e)$, is $2\pi \alpha$ if traversed along an upward triangle, and $2\pi \beta$ if along a downward triangle. For consistency with self-similarity, $\alpha = \beta$. 
Spectral decimation of the line bundle Laplacian on SG

- Analyzed in several physics papers in the 80s, most notably Ghez–Wang–Rammal–Pannetier–Bellissard ’88.
- Ruoyu Guo (Colgate ’19) is working on a careful analysis of spectral decimation of \( \Delta_\theta \) as part of his senior honors thesis.

Proposition (GWRPB ’88, C.–Guo ’18+)

Let \((\Delta^{(n)}_\theta, \alpha)\) denote the line bundle Laplacian with constant flux \(2\pi\alpha\) on the \(n\)th-level SG. Then there is spectral decimation from \((\Delta^{(n+1)}_\theta, \alpha)\) to \((\Delta^{(n)}_\theta, 4\alpha)\) with spectral decimation function
\[
R(z, \alpha) = 1 + \frac{A(z, \alpha)}{2|\Psi(z, \alpha)|},
\]
where
\[
A(z, \alpha) = -64z^4 + 256z^3 - 356z^2 - [6\cos(2\pi\alpha) - 200]z
\]
\[
+ 6\cos(2\pi\alpha) + \cos(4\pi\alpha) - 37,
\]
\[
\Psi(z, \alpha) = 8z^2 - (2e^{2\pi i(3\alpha)} + 4e^{2\pi i\alpha} + 16)z + 2e^{2\pi i(3\alpha)} + \frac{3}{2}e^{4\pi i\alpha} + 4e^{2\pi i\alpha} + \frac{15}{2}.
\]
An approximate spectrum: Initialize with points $\lambda_0, \alpha_0$ in $[0, 2] \times [0, 1]$. Let $(\lambda_n, \alpha_n) = R^o_n(\lambda_0, \alpha_0)$. We expect the spectrum to \textit{roughly} coincide with the Julia set of $R$. To get a picture of the filled Julia set, we keep points $(\lambda_0, \alpha_0)$ for which $|R^o_n(\lambda_0, \alpha_0)| \leq C$ for all $n$.

\textbf{Figure:} The “Hofstadter butterfly” on the Sierpinski gasket
1. Make the connection between $\sigma((\Delta_\theta, \alpha))$ and $\mathcal{J}(R)$ exact.

2. Establish properties of the spectrum, e.g.: Is it true that for every $\alpha \in [0, 1]$, $\sigma(\Delta_\theta, \alpha) = \sigma_{pp}(\Delta_\theta, \alpha)$? Also, the bottom (and top) of the spectrum $\lambda_{\min}(\alpha)$ ($\lambda_{\max}(\alpha)$) seems to be continuous in $\alpha$. Is this true?

3. The line-bundle version of Kirchhoff’s matrix-tree theorem [Forman ’93, Kenyon ’10]

\[
\det \Delta_\theta = \sum_{\text{CRSFs}} \prod_{\text{cycles}} 2 \left( 1 - \cos \left( \sum_{e \in \text{cycle}} \theta(e) \right) \right),
\]

where the sum runs over all cycle-rooted spanning forests on the graph.

Leads to probabilistic analysis of random spatial processes on fractals.
Results to come (in 2019)

Thank you!