Random walks, electric networks, moving particle lemma, and hydrodynamic limits

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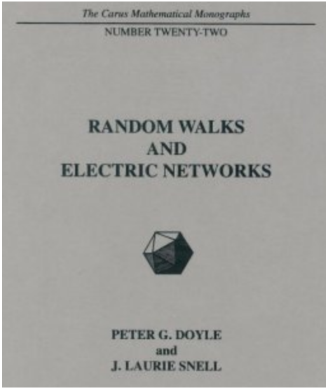


https://math.dartmouth.edu/~doyle/docs/walks/walks.pdf

Random walks and electric networks

- Let $G = (V, E)$ be a locally finite connected graph, and $\mathbf{c} = \{c_{xy}\}_{xy \in E}$ be the set of positive weights (conductances) endowed on E.
- The (symmetric) random walk process on the **weighted graph** (=**electric network**) $(G, \mathbf{c})$ is an irreducible Markov chain on V with transition probability

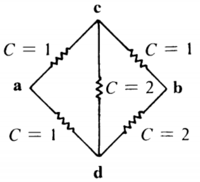
$$\mathbf{P}(x,y) = \begin{cases} c_{xy}/c_x, & \text{if } xy \in E, \\ 0, & \text{otherwise.} \end{cases} \qquad c_x := \sum_{z: xz \in E} c_{xz}.$$

- The RW process has $\pi(\cdot) \propto c(\cdot)$ as reversible (invariant) measure, and the associated Dirichlet energy is

$$\mathcal{E}^{\mathrm{RW}}(f) = \langle f, (\mathbf{I} - \mathbf{P}) f \rangle_\pi = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2, \quad f : V \to \mathbb{R}.$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{2}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix}$$

(The entries along each row must add up to 1.)



Conductances

Random walks and electric networks

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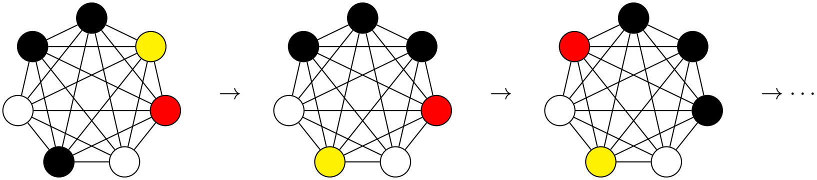
- **Effective resistance** between $A, B \subset V$:

$$R_{\mathrm{eff}}(A,B) = \sup \left\{ [\mathcal{E}^{\mathrm{RW}}(f)]^{-1} \,\middle|\, f : V \to \mathbb{R},\ f|_A = 1,\ f|_B = 0 \right\}$$

  In particular, if $A = \{x\}$ and $B = \{y\}$ we write $R_{\mathrm{eff}}(x,y)$. By definition,

$$[f(x) - f(y)]^2 \leq R_{\mathrm{eff}}(x,y) \mathcal{E}^{\mathrm{RW}}(f), \quad f : V \to \mathbb{R}.$$

  Also $R_{\mathrm{eff}} : V \times V \to \mathbb{R}_+$ is a metric on V.



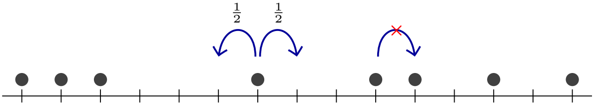
Overarching question: Can we study Markov processes involving MANY interacting "random walkers" on a weighted graph $(G, \mathbf{c})$?

Mathematical development started with Spitzer (on the integer lattice).
Mathematically tractable models:

1. **Exclusion process** (state space $\{0,1\}^V$): Particles perform RWs subject to the exclusion constraint that no two particles can occupy the same vertex at any time.
2. **Zero-range process** (state space $\mathbb{N}_0^V$): Particle at x jumps to neighboring y at rate depending on $\mathbf{P}(x,y)$ [jump] and the number of particles at x ONLY [zero-range kinetics].

Both models are associated with a conserved quantity—the total # of particles (unless additional dynamics or "reservoirs" are attached).

Particle system #1: Exclusion process

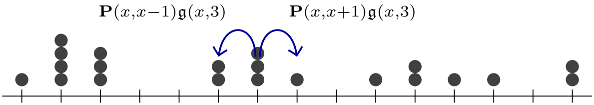


The **(symm.) exclusion process** on $(G, \mathbf{c})$ is a Markov chain on $\{0,1\}^V$ with generator

$$(\mathcal{L}^{\mathrm{EX}} f)(\eta) = \sum_{xy \in E} c_{xy} (\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $(\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

- Each product Bernoulli measure ν_α, $\alpha \in [0,1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure.
- Dirichlet energy: $\boxed{\mathcal{E}^{\mathrm{EX}}(f) = \frac{1}{2} \sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta)}$.



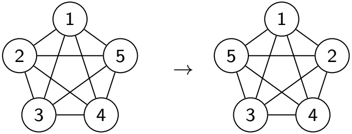
The **zero-range process** on $(G, \mathbf{c})$ is a Markov chain on $\mathbb{N}_0^V$ with generator

$$(\mathcal{L}^{\mathrm{ZR}} f)(\xi) = \sum_{(x,y)\in V^2} \mathbf{P}(x,y)\mathfrak{g}(x,\xi(x)) \left[f(\xi + \mathbf{1}_y - \mathbf{1}_x) - f(\xi)\right], \quad f : \mathbb{N}_0^V \to \mathbb{R}.$$

where $\mathbf{P}$ is an irreducible jump Markov matrix on V^2, and $\mathfrak{g} : V \times \mathbb{N}_0 \to \mathbb{R}_+$ is the kinetic rate, $\mathfrak{g}(x,0) = 0$ always.

- Invariant measure is a product one: $\mu(\xi) = \dfrac{1}{Z} \displaystyle\prod_{x\in V} \prod_{k=1}^{\xi(x)} \frac{\pi(x)}{\mathfrak{g}(x,k)}$, where π is the invariant measure for $\mathbf{P}$.
- Dirichlet energy: $\mathcal{E}^{\mathrm{ZR}}(f) = \langle f, -\mathcal{L}^{\mathrm{ZR}} f \rangle_\mu$.

Hierarchy of stochastic processes on a fixed graph

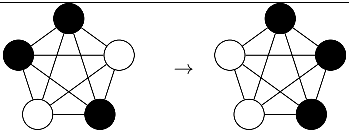


Interchange process $f : \{\text{Permutations on } V\} \to \mathbb{R}$

$$\mathcal{E}^{\mathrm{IP}}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw} [f(\eta^{zw}) - f(\eta)]^2 \, d\nu(\eta).$$

Reversible measure: uniform measure ν on $\{\text{Perms on } V\}$.

↓ PROJECTION ↓



Exclusion process $f : \{0,1\}^V \to \mathbb{R}$

$$\mathcal{E}^{\mathrm{EX}}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw} [f(\eta^{zw}) - f(\eta)]^2 \, d\nu_\alpha(\eta).$$

Reversible measure: product Bernoulli measure ν_α, $\alpha \in [0,1]$, $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for all $x \in V$.

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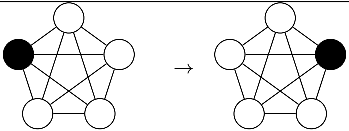


Random walk process $f : V \to \mathbb{R}$

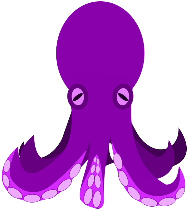
$$\mathcal{E}^{\mathrm{RW}}(f) = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2.$$

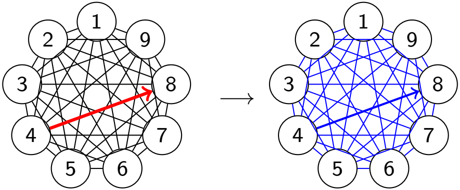
Reversible measure: $c(\cdot) = \sum_{w \sim \cdot} c_{w\cdot}$.

Aldous' spectral gap conjecture '92: Is $\lambda_2^{\mathrm{EX}}(G) = \lambda_2^{\mathrm{RW}}(G)$?

A projection argument easily leads to: $\lambda_2^{\mathrm{RW}}(G) \geq \lambda_2^{\mathrm{EX}}(G) \geq \lambda_2^{\mathrm{IP}}(G)$.
For the other direction, suffice to prove that $\lambda_2^{\mathrm{IP}}(G) \geq \lambda_2^{\mathrm{RW}}(G)$.

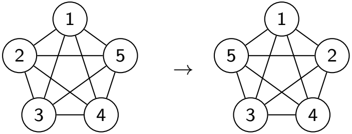
Random walks, electric networks, **moving particle lemma**, and hydrodynamic limits





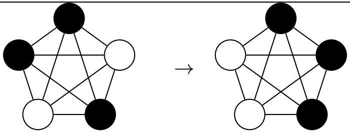
- Caputo, Liggett, and Richthammer, *J. Amer. Math. Soc.* (2010).
- C., *Electron. Commun. Probab.* (2017).

Hierarchy of stochastic processes on a fixed graph



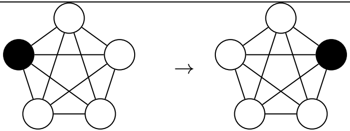
Interchange process $f : \{\text{Permutations on } V\} \to \mathbb{R}$
$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \le R_{\text{eff}}(x,y)\mathcal{E}^{\text{IP}}(f).$$
Moving particle lemma

↓ PROJECTION ↓



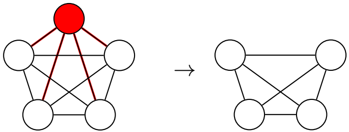
Exclusion process $f : \{0,1\}^V \to \mathbb{R}$
$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta) \le R_{\text{eff}}(x,y)\mathcal{E}^{\text{EX}}(f).$$
Moving particle lemma

↓ PROJECTION ↓



Random walk process $f : V \to \mathbb{R}$
$$[f(x) - f(y)]^2 \le R_{\text{eff}}(x,y)\mathcal{E}^{\text{RW}}(f).$$
Dirichlet principle
(Also a dual version involving flows: Thomson principle)

Energy inequalities
Does the MPL follow trivially from the Dirichlet principle? NO!
However, a common idea is **electric network reduction** (Schur complementation in linear algebra).



Idea: Remove vertices (and edges attached to them) without changing the effective conductance between any of the non-removed vertices.

- Suppose we remove the vertex $x \in V$ from $(G, \mathbf{c})$, as well as the edges attached to x.
  Call the reduced graph $G_x = (V_x, E_x)$.
  In the linear algebra language, we will reduce the Laplacian $\mathbf{L} = \mathbf{I} - \mathbf{P}$ to a new Laplacian $\mathbf{L}'$ (of one fewer dimension).
  This is attained by taking the Schur complement of the (x, x) block in $\mathbf{L}$:

$$\text{If } \mathbf{L} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{L}_{xx} \end{bmatrix}, \quad \text{then } \mathbf{L}' = \mathbf{X} - \mathbf{Y}(\mathbf{L}_{xx})^{-1}\mathbf{Z} = \mathbf{X} - \mathbf{Y}\mathbf{Z}. \qquad (\text{Recall } \mathbf{L}_{xx} = 1.)$$

- In component form, $\mathbf{L}'_{yz} = \mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}$ for $y, z \in V_x$.

  Since $\mathbf{L}^{(')}_{yz} = -p^{(')}_{yz} = -\frac{c^{(')}_{yz}}{c_y}$ whenever $y \neq z$, we see that the new conductances on E_x become

$$c'_{yz} = -c_y \mathbf{L}'_{yz} = -c_y(\mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}) = c_{yz} + \frac{c_{yx}c_{xz}}{c_x} =: c_{yz} + \tilde{c}_{yz}.$$

Example 1: Series Law

Let $c_{xy} = \alpha$ and $c_{xz} = \beta$.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} & 1 \end{bmatrix}.$$

Let $\mathbf{L}'$ be the Schur complement of the 1 block in $\mathbf{L}$:

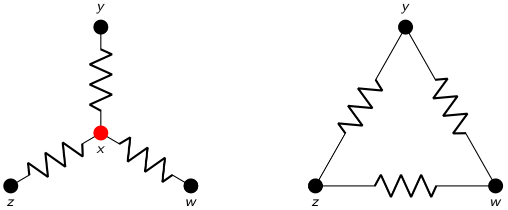
$$\mathbf{L}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \\ -\frac{\alpha}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

So $\mathbf{L}'_{yz} = -\frac{\beta}{\alpha+\beta}$. Since $c_y = \alpha$, we get $c'_{yz} = -c_y \mathbf{L}_{yz} = \dfrac{\alpha\beta}{\alpha+\beta}$, *i.e.*,

$$R'_{yz} = \frac{1}{c'_{yz}} = \frac{1}{\alpha} + \frac{1}{\beta} = R_{xy} + R_{xz}.$$

(Resistors in series ADD!)

Example 2: Y-Δ transform



Let $c_{xy} = \alpha$, $c_{xz} = \beta$, $c_{xw} = \gamma$, and $\sigma = \alpha + \beta + \gamma$.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \alpha/\sigma & \beta/\sigma & \gamma/\sigma & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -\alpha/\sigma & -\beta/\sigma & -\gamma/\sigma & 1 \end{bmatrix}.$$

$$\mathbf{L}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} -\alpha/\sigma & -\beta/\sigma & -\gamma/\sigma \end{bmatrix} = \frac{1}{\sigma} \begin{bmatrix} \beta+\gamma & -\beta & -\gamma \\ -\alpha & \alpha+\gamma & -\gamma \\ -\alpha & -\beta & \alpha+\beta \end{bmatrix}.$$

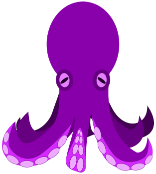
After a little more algebra we get

$$c'_{yz} = \frac{\alpha\beta}{\sigma}, \quad c'_{zw} = \frac{\beta\gamma}{\sigma}, \quad c'_{wy} = \frac{\gamma\alpha}{\sigma}.$$

(Anyone who studied electric circuits would find this familiar!)

Proof of Dirichlet's principle via network reduction

$$\mathcal{E}(f) = \sum_{zw\in E} c_{zw}[f(z)-f(w)]^2.$$



In going from G to the reduced graph G_x, energy is

- lost due to the removal of edges attached to x: amount $\sum_{y\in V_x} c_{xy}[f(x)-f(y)]^2$.
- gained due to the increased conductance on the non-removed edges: amount $\sum_{yz\in E_x} \tilde{c}_{yz}[f(y)-f(z)]^2$.

Proposition ("Octopus inequality" for electric network). For all $f: V\to\mathbb{R}$,

$$\sum_{y\in V_x} c_{xy}[f(x)-f(y)]^2 \geq \sum_{yz\in E_x} \tilde{c}_{yz}[f(y)-f(z)]^2,$$

Energy lost from removed edges $\geq$ *Energy gained from increased conductances*

where equality is attained iff $(\mathbf{L}f)(x)=0$.

Proof. An exercise in high school algebra.

Corollary. The Dirichlet energy is *monotone non-increasing* upon successive network reductions.

By carrying out network reduction one vertex at a time until two vertices z and y are left, we recover Dirichlet's principle: $\mathcal{E}(f)\geq c_{\rm eff}(z,y)[f(z)-f(y)]^2$.

Why the name "octopus"? The tentacular nature of removing of a vertex and its edges may remind you of an octopus. [*est.* Pietro Caputo.]

Octopus inequality & Aldous' spectral gap conjecture

Using the network reduction idea & delicately carrying out a series of Schur complementations, **Caputo–Liggett–Richthammer JAMS '10** proved for the **interchange process**:

Theorem (Octopus inequality, IP)

For all $f : \mathcal{S}_{|V|} \to \mathbb{R}$,

$$\int \sum_{y \in V_x} c_{xy} [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \geq \int \sum_{yz \in E_x} \tilde{c}_{yz} [f(\eta^{yz}) - f(\eta)]^2 \, d\nu(\eta).$$

Energy lost from removed edges $\geq$ *Energy gained from increased conductances*

This was the key inequality which resolved Aldous' '92 spectral gap conjecture:

$$\text{(OI)} \implies \lambda_2^{\mathrm{IP}}(G) \geq \lambda_2^{\mathrm{RW}}(G) \underset{+\text{proj.}}{\implies} \lambda_2^{\mathrm{IP}}(G) = \lambda_2^{\mathrm{EX}}(G) = \lambda_2^{\mathrm{RW}}(G).$$

- **MathSciNet review of CLR10, by L. Miclo:** "One leaves this beautiful paper with the dream that maybe a simpler proof could be found."
- Since then there have been attempts at simplifying the CLR proof, but to little avail.
- Also it was unclear if the octopus has any applications beyond resolving the spectral gap conjecture...

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RECENT DEVELOPMENTS — Applications of the octopus:

- **C. '17, Moving particle lemma**, used to carry out coarse-graining in the exclusion process towards proving hydrodynamic limits.
- **Alon–Kozma '18**, Improved estimates of mixing times of interchange process, energy level ordering in the Heisenberg ferromagnetic model.
  `arXiv:1811.10537`: "The first to use the octopus lemma for something new was Chen."
- (Related) **Hermon–Salez '18**: Analog of Aldous' spectral gap conjecture for the zero-range process, used to establish comparison theorems for two zero-range processes with the same kinetics on the same graph.

Moving particle lemma for interchange/exclusion [C. *ECP* '17]

Bounding the energy cost of swapping two particles at x and y in an interacting particle system by the **effective resistance** between x and y w.r.t. the random walk process.

Theorem (MPL, IP/EX)

$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \le R_{\rm eff}(x,y)\mathcal{E}^{\rm IP}(f), \quad f : \mathcal{S}_{|V|} \to \mathbb{R},$$
$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta) \le R_{\rm eff}(x,y)\mathcal{E}^{\rm EX}(f), \quad f : \{0,1\}^V \to \mathbb{R}.$$

Proof sketch.

- (OI) $\Leftrightarrow$ monotonicity of energy under 1-point network reductions. So reduce G successively until two vertices x, y are left, we get

$$\mathcal{E}^{\rm IP}(f) \ge \cdots \ge \frac{1}{2}\int c_{\rm eff}(x,y)[f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta). \quad \text{MPL for IP}$$

- To obtain the MPL for EX, use the projection of IP onto EX & disintegration of the uniform measure into orthonormal chambers with fixed particle number.

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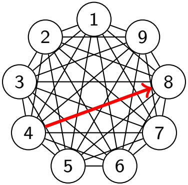

Conventional approach is to pick a single path connecting x and y and obtain the energy cost. [Guo–Papanicolaou–Varadhan '88, Diaconis–Saloff-Coste '93].

Works just fine on finite integer lattices, but does NOT always give optimal cost on general weighted graphs.

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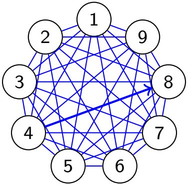
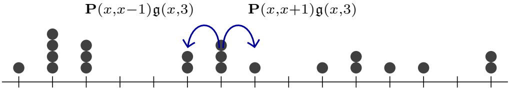


MPL bounds the energy cost by "optimizing electric flow over all paths connecting x and y."



$$(\mathcal{L}^{\rm ZR} f)(\xi) = \sum_{(x,y)\in V^2} \mathbf{P}(x,y)\mathfrak{g}(x,\xi(x))\left[f(\xi+\mathbf{1}_y-\mathbf{1}_x)-f(\xi)\right], \quad \text{inv. meas. } \mu.$$

Let $\Omega := \left\{\xi\in\mathbb{N}_0^V : \sum_{x\in V}\xi(x)=m\right\}$ and $\hat\Omega := \left\{\zeta\in\mathbb{N}_0^V : \sum_{x\in V}\zeta(x)=m-1\right\}$.

For each $f:\Omega\to\mathbb{R}$ and $\zeta\in\hat\Omega$, define $f_\zeta : V\to\mathbb{R}$ by $f_\zeta(x)=f(\zeta+\mathbf{1}_x)$.

Lemma. For all $f,g:\Omega\to\mathbb{R}$, $\mathcal{E}^{\rm ZR}_{(\mathbf{P},\mathfrak{g},m)}(f,g)=\sum_{\zeta\in\hat\Omega}\mu(\zeta)\langle f_\zeta,(\mathbf{I}-\mathbf{P})g_\zeta\rangle_\pi$. (Jump part decouples)

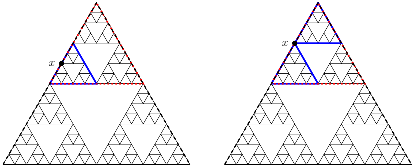
Theorem [Hermon–Salez '18]. For any two irred. jump matrices **P** and **Q**,

$$\min_{\substack{f:\Omega\to\mathbb{R}\\ f\neq 0}}\left\{\frac{\mathcal{E}^{\rm ZR}_{(\mathbf{P},\mathfrak{g},m)}(f,f)}{\mathcal{E}^{\rm ZR}_{(\mathbf{Q},\mathfrak{g},m)}(f,f)}\right\} = \min_{\substack{f:V\to\mathbb{R}\\ f\neq 0}}\left\{\frac{\langle f,(\mathbf{I}-\mathbf{P})f\rangle_{\pi_\mathbf{P}}}{\langle f,(\mathbf{I}-\mathbf{Q})f\rangle_{\pi_\mathbf{Q}}}\right\}.$$

Proposition [C.]. If **P** is associated to a symm. RW, then we have the **MPL**

$$\sum_{\zeta\in\hat\Omega}[f(\zeta+\mathbf{1}_y)-f(\zeta+\mathbf{1}_x)]^2\mu(\zeta)\le R_{\rm eff}(x,y)\mathcal{E}^{\rm ZR}_{(\mathbf{P},\mathfrak{g},m)}(f,f).$$

MPL & coarse-graining



For finite $\Lambda \subset V$, denote the **average density over** Λ by $\mathrm{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z \in \Lambda} \eta(z)$.
In the proof of the hydrodynamic limit for Markov processes, w/ generator $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$ on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

Replacement lemma

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{N \to \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t \left(\eta_s^N(x) - \mathrm{Av}_{B(x, \epsilon N)}[\eta_s^N] \right) ds \right| \right] = 0, \quad x \in V_N.$$

where

- $\{\eta_t^N : t \geq 0\}$ is the exclusion process generated by $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$, where $\mathcal{T}_N$ is the diffusive time acceleration factor.
- μ_N can be any measure on $\{0, 1\}^{V_N}$.
- $B(x, r)$ is a "ball" of radius r centered at x (in the graph metric).

MPL & coarse-graining

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Replacement lemma

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{N \to \infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t g(\eta_s^N)\, ds\right|\right] = 0, \text{ where } g(\eta) := \eta(x) - \mathrm{Av}_{B(x,\epsilon N)}[\eta],\ x \in V_N.$$

The usual method to control additive functionals of the EX process is to employ the entropy inequality, Jensen's inequality, and the Feynman-Kac formula:

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t g(\eta_s^N)\, ds\right|\right] \le \frac{H(\mu_N|\nu^N_{\rho(\cdot)})}{\kappa|V_N|} + \frac{1}{\kappa|V_N|}\sup_f \left\{\int g(\eta) f(\eta) d\nu^N_{\rho(\cdot)}(\eta) - \frac{\mathcal{T}_N}{\kappa|V_N|}\langle \sqrt{f}, -\mathcal{L}_N \sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\right\}$$

where

- $\rho(\cdot) \in \mathrm{dom}\mathcal{E}$ is a (possibly non)constant reference density profile.
- $H(\mu|\nu) = \int \log\left(\frac{d\mu}{d\nu}\right) d\mu$ is the relative entropy of μ w.r.t. ν, assumed to be $\mathcal{O}(|V_N|)$.
- $\kappa > 0$.
- The supremum is taken over all prob. densities f w.r.t. the product Bernoulli measure $\nu^N_{\rho(\cdot)}$.

MPL & coarse-graining

In the proof of the hydrodynamic limit for Markov processes, w/ generator $\mathcal{T}_N \mathcal{L}_N^{\rm EX}$ on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

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Assume for this discussion that $\rho(\cdot) = \rho$ constant. We wish to estimate

$$\int g(\eta)f(\eta)d\nu_\rho^N(\eta) - \frac{\mathcal{T}_N}{\kappa|V_N|}\langle\sqrt{f}, -\mathcal{L}_N\sqrt{f}\rangle_{\nu_\rho^N}$$

independent of f and the carré du champ

$$\mathcal{D}_N(\sqrt{f}, \nu_\rho^N) := \frac{1}{2}\int \sum_{zw\in E_N} c_{zw}\left(\sqrt{f(\eta^{zw})} - \sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta).$$

Using the Cauchy-Schwarz (Young) inequality and several elementary tricks, we get for any $A > 0$,

$$\int g(\eta)f(\eta)\,d\nu_\rho^N(\eta) \le \frac{1}{2|B|}\sum_{z\in B}\left\{\frac{A}{2}\int(\eta(z)-\eta(x))^2\left(\sqrt{f(\eta^{zx})}+\sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta)\right.$$

$$\left. + \frac{1}{2A}\int\left(\sqrt{f(\eta^{zx})} - \sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta)\right\}. \quad (B = B(x,\epsilon N))$$

MPL & coarse-graining

In the proof of the hydrodynamic limit for Markov processes, w/ generator $\mathcal{T}_N \mathcal{L}_N^{\rm EX}$ on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

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This last term needs to be bounded by something times the carré du champ

$$\mathcal{D}_N(\sqrt{f}, \nu_\rho^N) := \frac{1}{2}\int \sum_{zw\in E_N} c_{zw}\left(\sqrt{f(\eta^{zw})} - \sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta).$$

Use the MPL:

$$\begin{aligned}\frac{1}{2|B|}\sum_{z\in B}\int\left(\sqrt{f(\eta^{zx})} - \sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta) &\leq \frac{1}{|B|}\sum_{z\in B} R_{\rm eff}(z,x)\mathcal{D}_N(\sqrt{f}, \nu_\rho^N)\\ &\leq \mathrm{diam}_R(B)\mathcal{D}_N(\sqrt{f}, \nu_\rho^N),\end{aligned}$$

where $\mathrm{diam}_R(B)$ is the diameter of B in the resistance metric. $(B = B(x, \epsilon N))$

Assuming that $\dfrac{|V_N|}{\mathcal{T}_N}\mathrm{diam}_R(B)$ is bounded for all N—this holds for **resistance spaces** in general— we can then choose A wisely to bound the variational functional from above by an expression which tends to 0 in the limit. This proves the replacement lemma.

MPL & coarse-graining

In the proof of the hydrodynamic limit for Markov processes, w/ generator $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$ on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

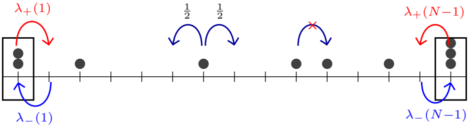
Replacement lemma

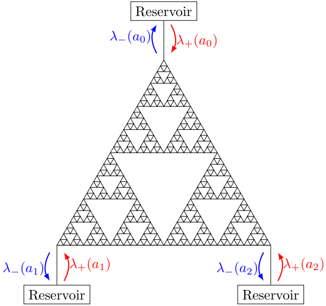
$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{N \to \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t g(\eta_s^N)\, ds \right| \right] = 0, \text{ where } g(\eta) := \eta(x) - \mathrm{Av}_{B(x,\epsilon N)}[\eta], \; x \in V_N.$$

- AFAIK this is the first time such an argument works on a non-lattice weighted graph, where translational invariance is absent.
- **Other usages of MPL:** Local 2-blocks estimate [C. '17]; 2nd-order Boltzmann-Gibbs principle for equilibrium density fluctuations [C. '19+].
- Another instance where one needs to prove such a replacement lemma in the absence of translational invariance: Studying non-equilibrium density fluctuations on $(\mathbb{Z}/N\mathbb{Z})^d$.

  Jara–Menezes '18 came up with their coarse-graining approach, called the "flow lemma," which utilizes mass distribution on the lattice, and is reminiscent of the **divisible sandpile** problem [Levine–Peres '09].

Random walks, electric networks, moving particle lemma, and **hydrodynamic limits**

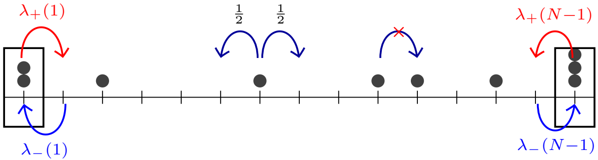




Scaling limits of empirical density in the boundary-driven SEP on the Sierpinski gasket

- LLN & CLT: Joint work with Patrícia Gonçalves (IST Lisboa), `arXiv:1904.08789`.
- LDP: Joint work with Michael Hinz (Bielefeld), preprint soon.

Adding reservoirs (Glauber dynamics) to the exclusion process



Designate a finite boundary set $\partial V \subset V$. For each $a \in \partial V$:

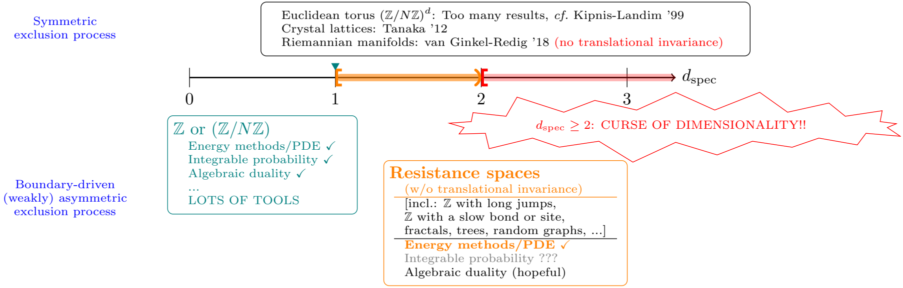
- At rate $\lambda_+(a)$, $\eta(a) = 0 \to \eta(a) = 1$ (birth).
- At rate $\lambda_-(a)$, $\eta(a) = 1 \to \eta(a) = 0$ (death).

Formally, $(\mathcal{L}_{\partial V}^{\rm boun} f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1-\eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)]$, $\quad f : \{0,1\}^V \to \mathbb{R}$, where

$$\eta^a(z) = \begin{cases} 1-\eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

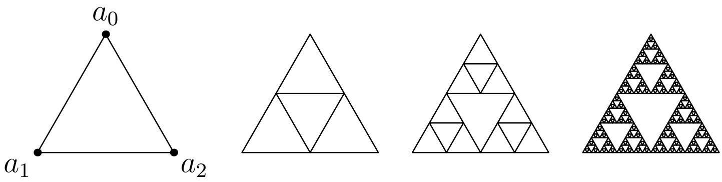
- 1D boundary-driven simple exclusion process: generator $N^2 \left(\mathcal{L}^{\rm EX}_{\{1,2,\cdots,N-1\}} + \mathcal{L}^{\rm boun}_{\{1,N-1\}} \right)$.
- Has been studied extensively for the past ~ 15 years: Hydrodynamic limits, fluctuations, large deviations, etc.
- Difficulties: $\#$ of particles is no longer conserved; the invariant measure is in general not explicit.

Extending the analysis to higher dims & with > 2 reservoirs?



- **Today's message:** On state spaces with spectral dimension $d_{\text{spec}} \in [1, 2)$, we have a path towards proving scaling limits of SSEP/WASEP w/o requiring translational invariance.
- **Open question:** Prove scaling limits of boundary-driven SSEP/WASEP on state spaces with $d_{\text{spec}} \geq 2$.

Boundary-driven exclusion process on the Sierpinski gasket



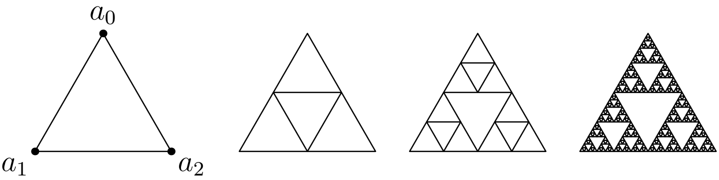
- Construction of **Brownian motion** with invariant measure m (the standard self-similar measure) as scaling limit of RWs accelerated by $\mathcal{T}_N = 5^N$.
  [Goldstein '87, Kusuoka '88, Barlow-Perkins '88]
- A **robust notion of calculus** on *SG* which in some sense mimics (but in many other senses differs from) calculus in 1D: Laplacian, Dirichlet form, integration by parts, boundary-value problems, etc.
  [Kigami, *Analysis on Fractals* '01; Strichartz, *Differential Equations on Fractals* '06]

What is the analog of "$\int_K |\nabla f|^2 \, dx$" in the fractal setting?

Corresponding domain—analog of $H^1(K, dx)$?

- A good model for rigorously studying (non)equilibrium stochastic dynamics with ≥ 3 boundary reservoirs.

Boundary-driven exclusion process on the Sierpinski gasket



- Construction of **Brownian motion** with invariant measure m (the standard self-similar measure) as scaling limit of RWs accelerated by $\mathcal{T}_N = 5^N$.
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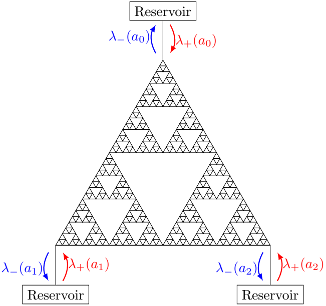
- A **robust notion of calculus** on SG, which in some sense mimics (but in many other senses differs from) calculus in 1D: Laplacian, Dirichlet form, integration by parts, boundary-value problems, etc.
  [Kigami, *Analysis on Fractals* '01; Strichartz, *Differential Equations on Fractals* '06]

$$\mathcal{E}(f) = \lim_{N\to\infty} \frac{5^N}{3^N} \sum_{xy\in E_N} (f(x) - f(y))^2, \quad f \in L^2(K, m).$$

$$\mathcal{F} := \left\{ f \in L^2(K, m) : \mathcal{E}(f) < +\infty \right\}.$$

- A good model for rigorously studying (non)equilibrium stochastic dynamics with ≥ 3 boundary reservoirs.

Boundary-driven exclusion process on the Sierpinski gasket



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Parameter $b > 0$ governs the inverse speed (relative to the bulk jump rate) at which the reservoir injects/extracts particles into/from the boundary vertices V_0.

Our main result in a nutshell [C.–Gonçalves '19]

A **phase transition** in the scaling limit of the particle density depending on the value of b, reflected by the different boundary conditions. The critical value of b is $\frac{5}{3}$.

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Hydrodynamic limit: a LLN result

Assume that sequence of probability measures $\{\mu_N\}_{N\geq 1}$ on $\{0,1\}^{V_N}$ is associated to a density profile $\varrho : K \to [0,1]$: for any continuous function $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta \in \{0,1\}^{V_N} \ : \ \left| \frac{1}{|V_N|} \sum_{x\in V_N} F(x)\eta(x) - \int_K F(x)\varrho(x)\, dm(x) \right| > \delta \right\} = 0.$$

Given the process $\{\eta_t^N : t \geq 0\}$ generated by $5^N \mathcal{L}_N^{\mathrm{bEX}}$, the **empirical density measure** π_t^N given by

$$\pi_t^N = \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x)\delta_{\{x\}}$$

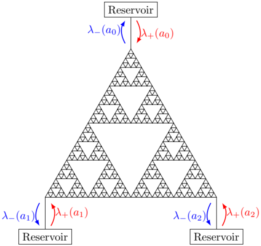
and for any test function $F : K \to \mathbb{R}$, we denote the integral of F wrt π_t^N by $\pi_t^N(F)$ which equals

$$\pi_t^N(F) = \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x)F(x).$$

Claim. The sequence $\{\pi_\cdot^N\}_N$ converges in the Skorokhod topology on $D([0,T],\mathcal{M}_+)$ to the unique measure $\pi_\cdot$ with $d\pi_\cdot(x) = \rho(\cdot, x)\, dm(x)$.
For any $t \in [0,T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x)F(x) - \int_K F(x)\rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

Hydrodynamic limit: a LLN result



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

$$\bar{\rho}(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)}$$

Theorem (Density hydrodynamic limit)

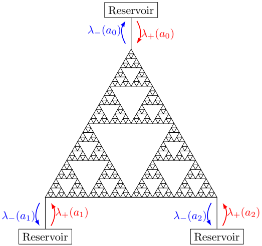
For any $t \in [0, T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with Dirichlet boundary condition if $b < \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ \rho(t,a) = \bar{\rho}(a), & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

Hydrodynamic limit: a LLN result



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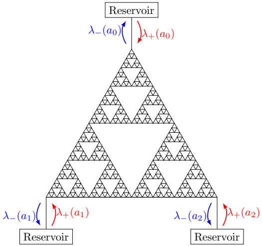
where ρ is the unique weak solution of the heat equation
with Neumann boundary condition if $b > \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3} \Delta \rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ (\partial^\perp \rho)(t,a) = 0, & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

Hydrodynamic limit: a LLN result



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$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \; : \; \left| \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with linear Robin boundary condition if $b = \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T], \; x \in K \setminus V_0, \\ (\partial^\perp \rho)(t,a) = -\lambda_\Sigma(a)(\rho(t,a) - \bar{\rho}(a)), & t \in (0,T], \; a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N\left(\left(\frac{2}{3}\Delta + \partial_s\right) F_s\right) ds$$
$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a)) F_s(a)\right] ds + o_N(1).$$

[Ingredient #1] **Analysis on fractals**
Convergence of discrete Laplacian to the continuous counterpart; normal derivatives at the boundary; integration by parts formula ... [Kigami '01, Strichartz '06].

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

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$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a)) F_s(a)\right] ds + o_N(1).$$

[Ingredient #1] **Analysis on fractals**
This part will produce the weak formulation of the heat equation.

Heuristics for hydrodynamics

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

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$$+\int_0^t \frac{3^N}{|V_N|}\sum_{a\in V_0}\left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a) - \bar\rho(a))F_s(a)\right]\,ds + o_N(1).$$

[Ingredient #2] **Analysis of the boundary term**

- $b > 5/3$: The first term dominates, should converge to $\int_0^t \frac{2}{3}\sum_{a\in V_0}\rho_s(a)(\partial^\perp F_s)(a)\,ds$
- $b = 5/3$: Both terms contribute equally, should converge to $\int_0^t \frac{2}{3}\sum_{a\in V_0}\left[\rho_s(a)(\partial^\perp F_s)(a) + \lambda_\Sigma(a)(\rho_s(a) - \bar\rho(a))F_s(a)\right]\,ds$
- $b < 5/3$: Impose $\rho_t(a) = \bar\rho(a)$ for all $a \in V_0$, should converge to $\int_0^t \frac{2}{3}\sum_{a\in V_0}\bar\rho(a)(\partial^\perp F_s)(a)$

Require a series of **replacement lemmas** — not trivial on state spaces without translational invariance!
[Thankfully, my MPL can be used to establish the replacement lemmas!]

Heuristics for hydrodynamics

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

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$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a))F_s(a) \right] ds + o_N(1).$$

$$\Big\downarrow N \to \infty$$

$$0 = \pi_t(F_t) - \pi_0(F_0) - \int_0^t \pi_s \left(\left(\frac{2}{3}\Delta + \partial_s \right) F_s \right) ds + (\text{boundary term})$$

[Ingredient #3] **Convergence of stochastic processes**

- Show that $\{\pi_\cdot^N\}_N$ is tight in the Skorokhod topology on $D([0,T], \mathcal{M}_+)$ via Aldous' criterion.
- Prove that any limit point $\pi_\cdot$ is absolutely continuous w.r.t. the self-similar measure m, with $\pi_t(dx) = \rho(t,x)\, dm(x)$, and $\rho \in L^2(0,T,\mathcal{F})$.
- Finally, prove ! of the weak solution to the heat equation to conclude ! of the limit point.

Density fluctuation field (at equilibrium): Heuristics

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)
The product Bernoulli measure ν_ρ^N with $\rho = \lambda_+/(\lambda_+ + \lambda_-)$ is stationary for the process.

Density fluctuation field (DFF) $\qquad \mathcal{Y}_t^N(F) = \dfrac{1}{\sqrt{|V_N|}} \displaystyle\sum_{x \in V_N} \left(\eta_t^N(x) - \rho\right) F(x)$

The corresponding Dynkin's martingale is

$$\mathcal{M}_t^N(F) = \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1)$$
$$+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma F(a)\right] ds,$$

which has QV

$$\langle M^N(F)\rangle_t = \int_0^t \frac{5^N}{|V_N|^2} \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} (\eta_s^N(x) - \eta_s^N(y))^2 (F(x) - F(y))^2 ds$$
$$+ \int_0^t \sum_{a \in V_0} \frac{5^N}{b^N |V_N|^2} \{\lambda_-(a)\eta_s^N(a) + \lambda_+(a)(1 - \eta_s^N(a))\} F^2(a) ds.$$

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)
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$$\begin{aligned}\mathcal{M}_t^N(F) &= \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1) \\ &+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma F(a) \right] ds,\end{aligned}$$

which, as $N \to \infty$, has the QV of a space-time white noise (with boundary condition)

$$\frac{2}{3} \cdot 2\rho(1-\rho) t \mathscr{E}_b(F), \quad \text{where } \mathscr{E}_b(F) = \mathcal{E}(F) + \lambda_\Sigma \sum_{a \in V_0} F^2(a) \mathbf{1}_{\{b = 5/3\}}$$

Density fluctuation field (at equilibrium): Heuristics

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)
The product Bernoulli measure ν_ρ^N with $\rho = \lambda_+/(\lambda_+ + \lambda_-)$ is stationary for the process.

Density fluctuation field (DFF) $$\mathcal{Y}_t^N(F) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \left(\eta_t^N(x) - \rho\right) F(x)$$

The corresponding Dynkin's martingale is

$$\mathcal{M}_t^N(F) = \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1)$$
$$+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma F(a) \right] ds,$$

We then argue that the test function $F \in \mathrm{dom}\Delta_b$ be chosen appropriate to each boundary condition such that the boundary term vanishes as $N \to \infty$.

$$\mathrm{dom}\Delta_b := \begin{cases} \{F \in \mathrm{dom}\Delta : F|_{V_0} = 0\}, & \text{if } b < 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = -\lambda_\Sigma F|_{V_0}\}, & \text{if } b = 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = 0\}, & \text{if } b > 5/3. \end{cases}$$

For technical reasons (in order to use Mitoma's tightness criterion) we use a smaller test function space $\mathcal{S}_b := \{F \in \mathrm{dom}\Delta_b : \Delta_b F \in \mathrm{dom}\Delta_b\}$, which can be made into a Frechét space.
Let S_b' be the topological dual of S_b.

Ornstein-Uhlenbeck equation with boundary condition

Definition (Ornstein-Uhlenbeck equation)

We say that a random element $\mathcal{Y}$ taking values in $C([0,T], \mathcal{S}_b')$ is a solution to the **Ornstein-Uhlenbeck equation** on K with parameter b if:

1. For every $F \in \mathcal{S}_b$,

$$\mathcal{M}_t(F) = \mathcal{Y}_t(F) - \mathcal{Y}_0(F) - \int_0^t \mathcal{Y}_s(\frac{2}{3}\Delta_b F)\, ds$$

$$\text{and} \quad \mathcal{N}_t(F) = (\mathcal{M}_t(F))^2 - \frac{2}{3} \cdot 2\rho(1-\rho)t\mathscr{E}_b(F)$$

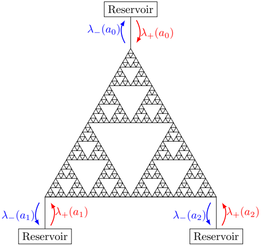
   are $\mathscr{F}_t$-martingales, where $\mathscr{F}_t := \sigma\{\mathcal{Y}_s(F) : s \leq t\}$ for each $t \in [0,T]$.

2. $\mathcal{Y}_0$ is a centered Gaussian $\mathcal{S}_b'$-valued random variable with covariance

$$\mathbb{E}_\rho^b[\mathcal{Y}_0(F)\mathcal{Y}_0(G)] = \rho(1-\rho)\int_K F(x)G(x)\, dm(x), \quad \forall F, G \in \mathcal{S}_b.$$

Moreover, for every $F \in \mathcal{S}_b$, the process $\{\mathcal{Y}_t(F) : t \geq 0\}$ is Gaussian: the distribution of $\mathcal{Y}_t(F)$ conditional upon $\mathscr{F}_s$, $s < t$, is Gaussian with mean $\mathcal{Y}_s(\tilde{\mathrm{T}}^b_{t-s}F)$ and variance $\int_0^{t-s} \frac{2}{3} \cdot 2\rho(1-\rho)\mathscr{E}_b(\tilde{\mathrm{T}}^b_r F)\, dr$, where $\{\tilde{\mathrm{T}}^b_t : t > 0\}$ is the heat semigroup associated with $\frac{2}{3}\mathscr{E}_b$.

O-U limit of equilibrium density fluctuations: a CLT result



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Equilibrium $\Leftrightarrow$ $\lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$.

Let $\mathbb{Q}_\rho^{N,b}$ be the probability measure on $D([0,T], \mathcal{S}_b')$ induced by the DFF $\mathcal{Y}_\cdot^N$ started from ν_ρ^N and boundary parameter b.

Theorem (CLT)

The sequence $\{\mathbb{Q}_\rho^{N,b}\}_N$ converges in distribution, as $N \to \infty$, to a unique solution of the Ornstein-Uhlenbeck equation with parameter b (as defined previously).

Key Lemma. $\tilde{\mathrm{T}}_t^b(\mathcal{S}_b) \subset \mathcal{S}_b$ for any $t > 0$. Enough to verify that $\tilde{\mathrm{T}}_t^b\left(L^1(K,m)\right) \subset \mathrm{dom}\Delta_b$, which can be shown using *e.g.* the Nash inequality (heat kernel upper bound).

The rest of the argument follows a martingale approach of Kipnis–Landim.

Density large deviations principle (Dirichlet case)

- $\mathbb{Q}^N$: Law of the Markov process generated by $5^N \mathcal{L}_N^{\mathrm{bEX}}$, with $b = 1$.
- $\mathcal{M}_+$: Space of nonnegative Borel measures on K.
- $\mathcal{F}_0 := \{f \in \mathcal{F} : f|_{V_0} = 0\}$.

Theorem (Density LDP: rate $|V_N| \sim \frac{3}{2}3^N$ with good rate function I_0)

For each closed set $\mathcal{C}$ and each open set $\mathcal{O}$ of the Skorokhod space $D([0,T], \mathcal{M}_+)$, endowed with the Skorokhod topology of weak convergence of measures w.r.t. the Dirichlet problem,

$$\limsup_{N\to\infty} \frac{1}{|V_N|} \log \mathbb{Q}^N[\mathcal{C}] \leq - \inf_{\pi \in \mathcal{C}} I_0(\pi), \quad \liminf_{N\to\infty} \frac{1}{|V_N|} \log \mathbb{Q}^N[\mathcal{O}] \geq - \inf_{\pi \in \mathcal{O}} I_0(\pi).$$

Let $\mathcal{M}_{+,1} = \{\mu \in \mathcal{M}_+ \mid \mu(dx) = \rho(x)\, m(dx),\ 0 \leq \rho \leq 1\ m\text{-a.e.}\}$ and

$$D_{+,1,\mathcal{E}}[0,T] := \left\{\pi \in D([0,T], \mathcal{M}_{+,1}) \mid \pi(t,dx) = \rho(t,x)\, m(dx),\ \rho \in L^2(0,T,\mathcal{F})\right\}.$$

$I_0(\pi) < \infty \Longleftrightarrow \pi \in D_{+,1,\mathcal{E}}[0,T]$; then $\exists H \in C([0,T], \Delta^{-1}(\mathcal{F}_0)) \cap C^1((0,T), \Delta^{-1}(\mathcal{F}_0))$ s.t.

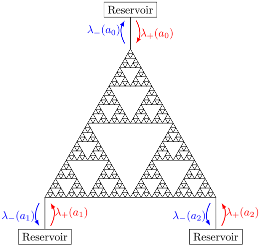
$$I_0(\pi) = \frac{1}{2}\int_0^T \int_K \rho(t,x)\,(1-\rho(t,x))\, d\Gamma(H_t)\, dt\,.$$

where $d\Gamma(F)$ is the **energy measure** on K defined via $\mathcal{E}(F) = \int_K d\Gamma(F)$.

N.B.: For nonconstant $F \in \mathrm{dom}\mathcal{E}$, $d\Gamma(F) \perp dm$. This is a source of major technical difficulties.

Technical Remark. The topology we use guarantees that $D_{+,1,\mathcal{E}}[0,T]$ is closed.

A sneak preview of upcoming series of works, and **Thank you!**



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Symmetric exclusion process with **slowed** boundary on the Sierpinski gasket

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)

- (Non)equilibrium density hydrodynamic limit (DRN✓) [C.–Gonçalves '19]
- Ornstein-Uhlenbeck limit of equilibrium density fluctuations (DRN✓). [C.–Gonçalves '19]
- Large deviations principle for the (non)equilibrium density (D✓) [C.–Hinz '19]
- Hydrostatic limit, scaling limit of nonequilibrium density fluctuations (D in progress). [C.–Franceschini–Gonçalves–Menezes '19+] $\rightarrow$ *careful study of two-particle correlations*
- **More in the pipeline:**

  Motion of the tagged particle (a fractional BM on the gasket?).

  Add (suitably rescaled) weak asymmetry to the jump rate, prove that the equilibrium density fluctuations converges (subsequentially) to a stochastic Burgers equation [C. '19+]