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Laplacian growth and sandpiles on the Sierpiński gasket: limit shape universality and exact solutions. (English summary)

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Laplacian growth models are, according to the authors' terminology, a class of growth processes on a connected, locally finite, undirected graph $G=(V(G),E(G))$ governed by the so-called *combinatorial graph Laplacian,* defined by the symmetric matrix $\Delta_G(x,y)$, $x,y\in V(G)$, defined by $\Delta_G(x,y)=-\deg(x)$ if $x=y$ and if $x\neq y$, $\Delta_G(x,y)$ is the number of edges in $E(G)$ that connect x and y.

An *abelian sandpile* on G is a mapping $s\colon V(G)\to\mathbb{Z}$, where $s(x)$ can be seen as the number of sand grains or chips at the vertex x. If $s(x)\geq\deg(x)$ then x is unstable and *topples,* meaning that $\deg(x)$ chips move from x to its neighboring vertices along each of the edges connecting with x, so that s transforms in a new sandpile $s'(y)=s(y)+\Delta_G(x,y)$. If $s(x)\leq\deg(x)-1$, then the sandpile stabilizes. It is easy to see that if a sandpile stabilizes through some toppling process, the outcome of the process does not depend on the ordering of the vertices chosen in the toppling process [see L. Levine and Y. Peres, Bull. Amer. Math. Soc. (N.S.) **54** (2017), no. 3, 355–382 (Lemma 1.1); MR3662912], which explains the term *abelian.*

Another related growth process is the *rotor-router aggregation model* in a graph. Here each vertex in the graph is endowed with a *rotor* or arrow that aims periodically at its connecting vertices. The rotor is simple if it visits all these vertices exactly once per period. A *non-random walk* is then defined, in which a walker moves at each step from the vertex she occupies to the neighboring vertex aimed at by the rotor, which changes then its target to the next position. Two more growth processes studied in this paper are the *internal diffusion-limited aggregation* (IDLA), a random walk in the graph, and the divisible sandpiles, a non-integer and more tractable version of abelian sandpiles (see details in the reviewed text).

A valuable goal in the sandpile community is to reject or confirm the so-called *limit shape universality conjecture,* which says that in a given state space (a graph) the *clusters* (whose definitions vary with the model) of all four models (IDLA, rotor-router aggregation, divisible sandpiles and abelian sandpiles) have the same limit shape. For instance, in the case of $\mathbb{Z}^d$ as state space, IDLA, rotor router and divisible sandpiles have Euclidean balls as limit shapes (and then the goal is to determine the asymptotics of the radius) but the limit shape of the abelian sandpile in $\mathbb{Z}^2$ seems to be a polygon, so the conjecture fails in this case.

The results in this work stand for growth models in the normalized graph of the Sierpiński gasket (SG): Let G_0 be a graph with vertices, $V(G_0)$, at the three corners of an unit equilateral triangle, one of which is at the origin o, and with edges, $E(G_0)$, the three sides of that triangle. Consider also the set mapping $\Psi(A)=\bigcup_{i=1}^3\psi_i(A)$, $A\subset\mathbb{R}^2$ where the $\psi_i's$ are the homotheties in the plane with fixed points at $V(G_0)$ and contraction ratios 0.5. For $k\in\mathbb{N}$, form the *prefractal* $\Psi^k(G_0)$ and renormalize to the graph $G_k=2^k\Psi^k(G_0)$, which has edges of unit length and to which o always belongs. Then $\mathrm{SG}=\bigcup_{i=1}^\infty G_k$.

Theorem 1 gives the limit shape of the rotor-router in SG. Here m walkers are launched at the origin o, each one of which walks until he arrives to a previously unvisited site.

The rotors at all sites are simple. $\mathcal{R}(m)$ and $\sigma(m)$ are respectively the sets of sites visited at least twice and the set of sites occupied by some walker. Let $B_x(r)$ be the closed ball of SG with respect to the graph metric, centered at $x \in \mathrm{SG}$ and with radius r; write $|A|$ for the cardinality of a set $A \subset \mathrm{SG}$; let $b_n = |B_o(n)| - \frac{1}{2}|\partial_I B_o(n)|$, where $\partial_I A$ denotes the set of points in $A \subset \mathrm{SG}$ connected to some vertex in $\mathrm{SG} - A$ and let $n_m = \max\{k \geq 0 : b_k \leq m\}$. Then, for all $m \in \mathbb{N}$ we have

$$B_o(n_m - 2) \subset \mathcal{R}(m) \subset B_o(n_m) \text{ and } B_o(n_m - 1) \subset \sigma(m) \subset B_o(n_m + 1).$$

Theorem 2 gives the shape of the clusters $S(m)$ of visited sites and $A(m)$ of sites that topple for an initial configuration of m chips at the origin o in the abelian sandpile model. We have

$$B_o(r_m - 1) \subset A(m) \subset B_0(r_m) = S(m).$$

If we define $r(x) = r_{\lfloor x \rfloor}$ for $x > 0$ then the radius r_m is given by the function

$$r(x) = x^{\frac{1}{d_H}} \left[\mathcal{G}(\log x) + o(1)\right] \text{ as } x \to \infty,$$

where $\mathcal{G}$ is a non-constant $(\log 3)$-periodic function with finitely many well-defined discontinuities (that cause *sandpile avalanches*) within each period and $d_H = \frac{\log 3}{\log 2}$ is the Hausdorff dimension of the Sierpiński gasket. This improves the result $r_m = \mathcal{O}(m^{\frac{1}{d_H}})$ in [S. Fairchild et al., "The abelian sandpile model on fractal graphs", preprint, arXiv:1602.03424].

Theorems 1 and 2 together with results in [J. P. Chen et al., in *Analysis and geometry on graphs and manifolds*, 126–155, Lond. Math. Soc. Lect. Note Ser., 461, Cambridge Univ. Press, Cambridge, 2020] and [W. Huss and E. Sava-Huss, Fractals **27** (2019), no. 3, 1950032; MR3957187] (Propositions 1.1 and 1.2 in the text) establish that the four models on SG, launched from o, fill balls in the graph metric centered at o (limit shape universality on SG, Theorem 3). SG is the first non-tree state space for which limit shape universality has been proven for all four aggregation models mentioned above.

Theorems 4, 5 and 6 are stages in the proof of Theorem 2. The proof combines ideas and tools from analysis, probability, algorithms, combinatorics, algebra and geometry.

In addition to the highlights given by the authors in the abstract, it is worth mentioning the relevant role played by the arrowhead Sierpiński curve in the proof of these results.

{Reviewer's comments. The first sandpile aggregation model was invented in 1987 by the physicists Bak, Tang and Wiesenfeld. Since then there has been much growth in the literature in this field, in which group theory, potential theory, partial (free boundary) differential equations, combinatorics, number theory and statistical physics, among others, interact. According to the reviewed text, fractal geometry can be added to this enumeration. The reader can get a hint of the richness of connections through the list of Mathematics Subject Classifications of related fields that the authors provide. This complex paper adds valuable results to the field.} *Manuel Morán*