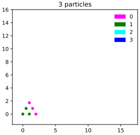
# Laplacian growth & sandpiles on the Sierpinski gasket

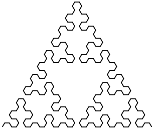
## Limit shape universality, fluctuations & beyond

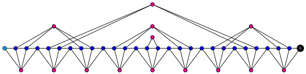
Joe P. Chen

Department of Mathematics
Colgate University
Hamilton, NY

Probability & Statistical Physics Seminar
Department of Mathematics
The University of Chicago
April 26, 2019







# Executive summary

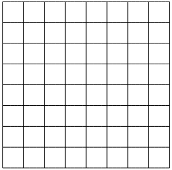
This talk gives a unified treatment of the following 3 papers:

1. **Internal DLA on Sierpinski gasket graphs.**
   JPC, Wilfried Huss, Ecaterina Sava-Huss, and Alexander Teplyaev.
   `arXiv:1702.04017`. To appear in "Analysis & Geometry on Graphs & Manifolds," London Mathematical Society Lecture Notes, Cambridge University Press (2019+).
2. **Divisible sandpiles on Sierpinski gasket graphs.**
   Wilfried Huss and Ecaterina Sava-Huss.
   `arXiv:1702.08370`. *Fractals* (2019).
3. **Laplacian growth & sandpiles on the Sierpinski gasket: limit shape universality & exact solutions.**
   JPC and Jonah Kudler-Flam.
   `arXiv:1807.08748`. Under review at *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* (2019+).

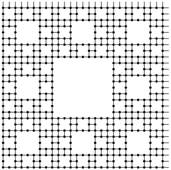
Also many thanks to:
Lionel Levine*, Bob Strichartz (and his REU students), LEGO, and Super Mario Bros.

*Lionel suggested this problem to me in 2012.

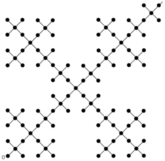
# Laplacian growth on lattices and graphs



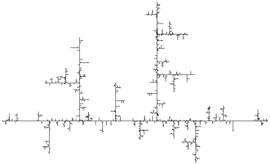
Square lattice



Sierpinski carpet

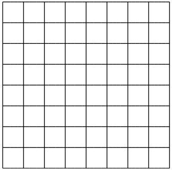


Vicsek tree

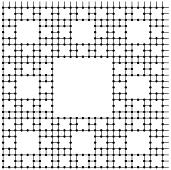


Random dendrite [by David Croydon]

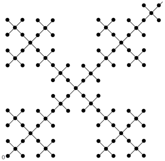
# Laplacian growth on lattices and graphs



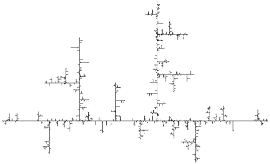
Square lattice



Sierpinski carpet



Vicsek tree



Random dendrite [by David Croydon]

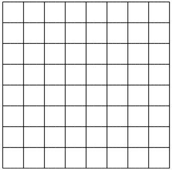
**Internal diffusion-limited aggregation (IDLA)**

Fix a distinguished vertex $o$.

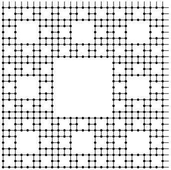
Set $\mathcal{I}(0) = \emptyset$.

For $n \geq 1$, define inductively $\mathcal{I}(n) := \mathcal{I}(n-1) \cup \{X^{(n)}_{\tau(\mathcal{I}(n-1)^c)}\}$, where $X^{(i)}$ are i.i.d. **random walks** started from $o$, and $\tau(A)$ is the first hitting time of $A$.

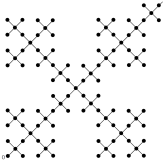
# Laplacian growth on lattices and graphs



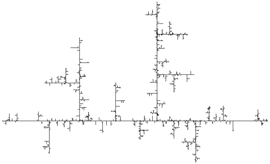
Square lattice



Sierpinski carpet

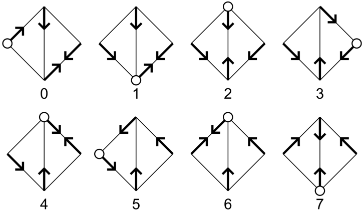


Vicsek tree

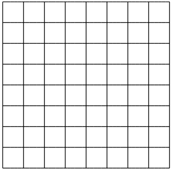


Random dendrite [by David Croydon]

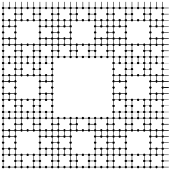
**Rotor walk** (derandomized random walk)



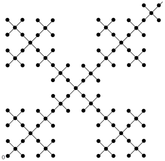
# Laplacian growth on lattices and graphs



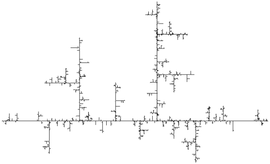
Square lattice



Sierpinski carpet



Vicsek tree



Random dendrite [by David Croydon]

**Rotor-router aggregation**

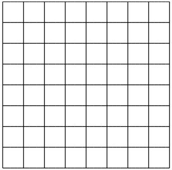
Fix a distinguished vertex $o$.

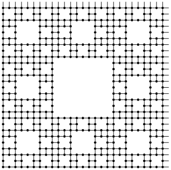
Set $\mathcal{R}(0) = \emptyset$.

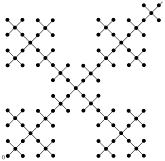
For $n \geq 1$, define inductively $\mathcal{R}(n) := \mathcal{R}(n-1) \cup \{Y^{(n)}_{\tau(\mathcal{R}(n-1)^c)}\}$, where $Y^{(i)}$ are **rotor walks** started from $o$, and $\tau(A)$ is the first hitting time of $A$.

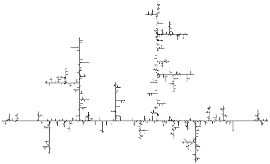
[Note that the rotor environment evolves in time!]

# Laplacian growth on lattices and graphs

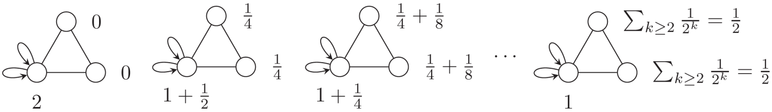

Square lattice


Sierpinski carpet

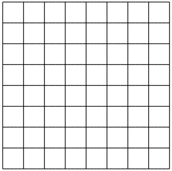

Vicsek tree


Random dendrite [by David Croydon]

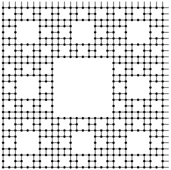
## Divisible sandpiles



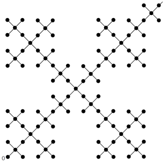
# Laplacian growth on lattices and graphs



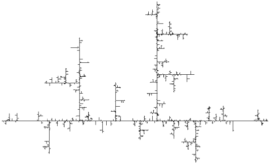
Square lattice



Sierpinski carpet

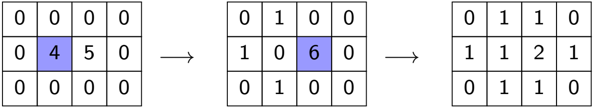


Vicsek tree



Random dendrite [by David Croydon]

**Abelian sandpiles**



|   |   |   |   |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 4 | 5 | 0 |
| 0 | 0 | 0 | 0 |

$\longrightarrow$

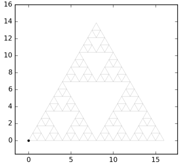
|   |   |   |   |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 6 | 0 |
| 0 | 1 | 0 | 0 |

$\longrightarrow$

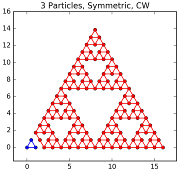
|   |   |   |   |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 1 | 1 | 2 | 1 |
| 0 | 1 | 1 | 0 |

# Questions of interest

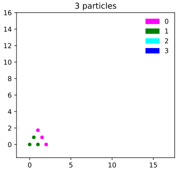
IDLA



Rotor-router



Abelian sandpiles



- Characterize the **limit shapes** (and fluctuations about the scaling limit) in each of the models.
- Fix a locally finite graph, and run all the growth models starting from $o$. Do the limit shapes coincide? (**Limit shape universality** [Levine-Peres '17])
- **Abelian sandpile model**: can also study the sandpile patterns! Does there exist a limit sandpile pattern? Other observables?

# Limit shapes for Laplacian growth & sandpiles

**$\mathbb{Z}^d$ $(d \geq 2)$** For all models: Launch $|B_o(n)|$ chips from $o$.

| *Model* | *Shape theorem/conjecture* |
|---|---|
| IDLA | In/out-radius $\begin{cases} n \pm \mathcal{O}(\log n), & d=2 \\ n \pm \mathcal{O}(\sqrt{\log n}), & d \geq 3 \end{cases}$ [α,β,γ,δ] |
| Rotor-router aggregation | In-radius $n - c\log n$, out-radius $n + c'\log n$ [κ,ℓ] |
| Divisible sandpiles | In-radius $n-c$, out-radius $n+c'$ [κ] |
| Abelian sandpiles | $(d=2)$ Limit shape closer to a dodecagon than Euc ball [κ]<br>Rigorous upper/lower estimates available (with a gap) [κ,ι] |

[α] Lawler–Bramson–Griffeath '92

[β] Lawler '95

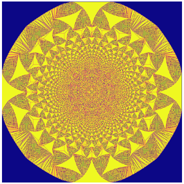
[γ] Asselah–Gaudillière '13 (2x)

[δ] Jerison–Levine–Sheffield '13, '14

[κ] Levine–Peres '09

[ℓ] Levine–Peres '17

[ι] Fey–Levine–Peres '10

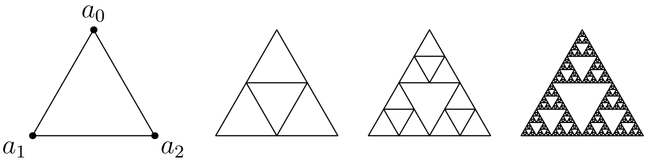


Levine–Peres, "Laplacian growth, sandpiles, and scaling limits." *Bull. Amer. Math. Soc.* (2017).

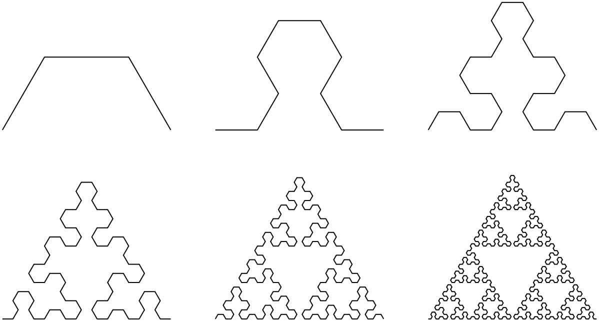
Not all 4 models have the same limit shape on $\mathbb{Z}^d$, $d \geq 2$.

# Sierpinski gasket (and two possible approximations)

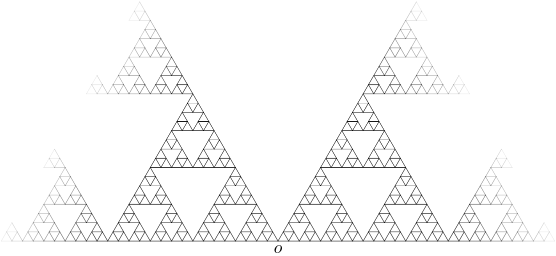
**The usual approximation**



**Sierpinski arrowhead curve** (space-filling)



# What makes the Sierpinski gasket so special?



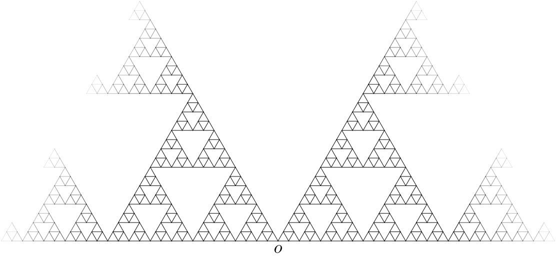
**Geometry**

- Self-similarity
- Discrete scale invariance
- Finite ramification: connected components separated by cut points
- Local ($D_3$) symmetries
- Special properties of spheres (centered at $o$)

**Analysis & Probability** [Kusuoka, Barlow, Perkins, Kigami, Strichartz, ...]

- Robust understanding of potential theory: random walk estimates, Green's function.

# What makes the Sierpinski gasket so special?



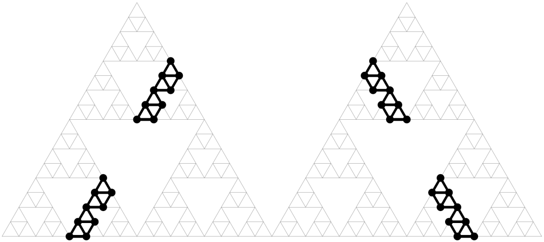
**Geometry**

- Self-similarity ⇒ Volume-doubling
- Discrete scale invariance ⇒ (arithmetic) renewal theorem applies
- Finite ramification: connected components separated by cut points ⇒ spatial “independence”
- Local ($D_3$) symmetries ⇒ Symmetries in Laplacian growth
- Special property of spheres (centered at $o$) ⇒ Sharp error control of shape boundary

**Analysis & Probability** [Kusuoka, Barlow, Perkins, Kigami, Strichartz, ...]

- Robust understanding of potential theory: random walk estimates, Green's function.
  ⇒ Harmonic measure is (nearly) uniform on spheres

# Special property of spheres in *SG*

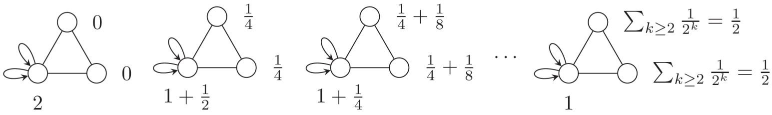


Inner boundary of $A$: $\partial_I A := \{x \in A : \exists y \in A^c,\ x \sim y\}$.

**Lemma.** For every $k \geq 1$, $\mathrm{b}_k := |B_o(k)| - \frac{1}{2}|\partial_I B_o(k)| = |B_o(k-1)| + |\partial_I B_0(k-1)|$.

# Odometer, least action principle

For divisible sandpiles



- Odometer function $u : V \to \mathbb{R}_+$.
  $u(x)$ counts the total amount of mass emitted from vertex $x$.
- Initial mass configuration $\mu_0$, odometer $u \Rightarrow$ get final mass configuration $\mu = \mu_0 + \Delta u$.
- Least action principle

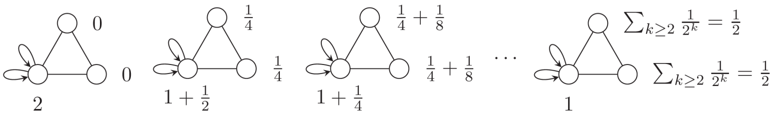
$$u(x) = \inf\{w(x) \mid w : V \to \mathbb{R}_+ \text{ satisfies } \mu_0 + \Delta w \leq 1\}$$

  A discrete obstacle problem. Works for any graph, but may not be easy to solve in practice.
- Thanks to the abelian property, there is an easier approach.

**Lemma.** Let $u_* : V \to \mathbb{R}_+$, $\mathcal{A}_* := \{x \in V : u_*(x) > 0\}$ and $\mu_* := \mu_0 + \Delta u_*$. If $\mathcal{A}_*$ is finite; $\mu_*(x) = 1$ for all $x \in \mathcal{A}_*$; and $\mu_* \leq 1$, then $u = u_*$.

- Analogous versions of odometer and LAP for: rotor-router aggregation (change $\Delta$ to the stack Laplacian $\Delta_\rho$), abelian sandpiles (functions $w$ must be $\mathbb{Z}$-valued).

# Divisible sandpile shape theorem on $SG$



## Theorem (Huss–Sava-Huss '17)

*Starting from* $\mu_0 = \mathrm{b}_n \mathbf{1}_o$*, the final divisible sandpile configuration is given by*

$$\mu(x) = \begin{cases} 1, & \text{if } x \in B_o(n) \setminus \partial_I B_o(n), \\ 1/2, & \text{if } x \in \partial_I B_0(n), \\ 0, & \text{otherwise.} \end{cases}$$

*The sandpile cluster* $S(\mathrm{b}_n)$ *equals* $B_o(n-1)$.

## Corollary

*For any* $m \geq 0$*, let* $n_m = \max\{k \geq 0 : \mathrm{b}_k \leq m\}$*. Then the sandpile cluster* $S(m)$ *on* $SG$ *with initial mass* $m$ *at* $o$ *satisfies* $B_o(n_m - 1) \subset S(m) \subset B_o(n_m)$.

# Using divisible sandpiles to analyze RRA & IDLA

**Heuristic.** Using the divisible sandpile shape/odometer, one can gain good control of shape/odometer for RRA & IDLA.

How is this heuristic used in practice?

# Using divisible sandpiles to analyze RRA & IDLA

**Heuristic.** Using the divisible sandpile shape/odometer, one can gain good control of shape/odometer for RRA & IDLA.

How is this heuristic used in practice?

## An exact fast simulation algorithm [Friedrich-Levine, *Random Structures Algorithms* '13]

1. Input an approximate odometer $u_1$, get $\sigma_1 = \sigma_0 + \Delta_\rho u_1$.
2. Correction #1: Fire hills and unfire holes in $\sigma_1$, return $\sigma_2$.
3. Correction #2: Reverse cycle-popping in $\sigma_2$ [*cf.* Wilson's algorithm '96], return correct config/odometer.

*Proof.* Least action principle.

*Note.* $\Delta_\rho$ is the **stack Laplacian** which depends on the rotor mechanism $\rho$, and is in general a nonlinear operator.

# Using divisible sandpiles to analyze RRA & IDLA

## An exact fast simulation algorithm [Friedrich-Levine, *Random Structures Algorithms* '13]

1. Input an approximate odometer $u_1$, get $\sigma_1 = \sigma_0 + \Delta_\rho u_1$.
2. Correction #1: Fire hills and unfire holes in $\sigma_1$, return $\sigma_2$.
3. Correction #2: Reverse cycle-popping in $\sigma_2$ [*cf.* Wilson's algorithm '96], return correct config/odometer.

*Proof.* Least action principle.

*Note.* $\Delta_\rho$ is the **stack Laplacian** which depends on the rotor mechanism $\rho$, and is in general a nonlinear operator.

- Dramatic speed-up in simulating RRA & IDLA (vs. brute-force inductive aggregation).
- The closer $u_1$ is to the correct odometer, the fewer corrections are needed.
- Luckily for us: When choosing $u_1$ to be the divisible sandpile odometer on *SG*, the algorithm works to prove a sharp rotor-router shape theorem on *SG*!

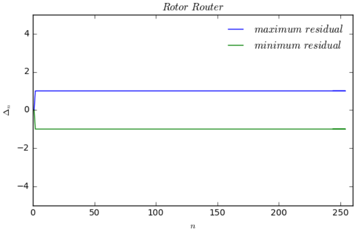
# Rotor-router shape theorem on *SG*

## Theorem (C.–Kudler-Flam '18)

*Let* $n_m = \max\{k \geq 0 : \mathsf{b}_k \leq m\}$*. Then for any periodic simple rotor mechanism,*

$$B_o(n_m - 2) \subset \mathcal{R}(m) \subset B_o(n_m), \qquad \forall m \in \mathbb{N}.$$

Numerics confirms that the bounds are sharp.



# Rotor-router shape theorem on *SG*

### Theorem (C.–Kudler-Flam '18)

*Let* $n_m = \max\{k \geq 0 : \mathrm{b}_k \leq m\}$*. Then for any periodic simple rotor mechanism,*

$$B_o(n_m - 2) \subset \mathcal{R}(m) \subset B_o(n_m), \qquad \forall m \in \mathbb{N}.$$

*Proof idea.* Start RRA with $\mathrm{b}_n$ particles. Use the divisible sandpile odometer $u_n^{\mathrm{DS}}$ as the input odometer into the F-L algorithm, then correct the errors, which takes place predominantly on the boundary.

$$\Delta_\rho u_n^{\mathrm{DS}}(x) \in \begin{cases} \{0,1,2\}, & \text{if } x \in S_o(n) \setminus \partial_I B_o(n), \\ \{0,1\}, & \text{if } x \in \partial_I B_o(n), \\ \{0\}, & \text{if } x \notin B_0(n). \end{cases}$$

Here $S_o(n) = B_o(n) \setminus B_o(n-1)$.

# Rotor-router shape theorem on *SG*

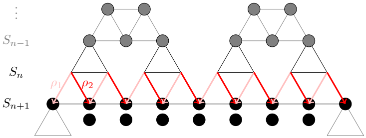
## Theorem (C.–Kudler-Flam '18)

*Let* $n_m = \max\{k \geq 0 : \mathrm{b}_k \leq m\}$. *Then for any periodic simple rotor mechanism,*

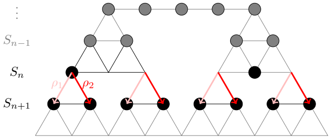
$$B_o(n_m - 2) \subset \mathcal{R}(m) \subset B_o(n_m), \qquad \forall m \in \mathbb{N}.$$

*Proof idea.* Start RRA with $\mathrm{b}_n$ particles. Use the divisible sandpile odometer $u_n^{\mathrm{DS}}$ as the input odometer into the F-L algorithm, then correct the errors, which takes place predominantly on the boundary.
An inductive proof in 2 acts:

1. "Fill the bulk": Rotor particles cannot overrun $B_o(n+1)$.
2. "Pull the marionette:" every vertex in $B_o(n)$ must be occupied.

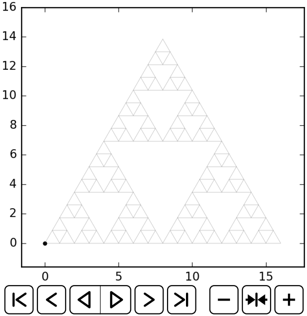


(a) *n* odd



(b) *n* even

# IDLA shape theorem on *SG*



**Theorem (C.–Huss–Sava-Huss–Teplyaev '17)**

*For every* $\epsilon > 0$,

$$B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon))$$

*holds for all sufficiently large* $n$*, with probability* 1.

# IDLA shape theorem on $SG$

### Theorem (C.–Huss–Sava-Huss–Teplyaev '17)

*For every* $\epsilon > 0$,

$$B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon))$$

*holds for all sufficiently large* $n$*, with probability* 1*.*

**Inner bound** *proof idea.*

- Establish a mean value inequality for the Green's function killed upon exiting $B_o(n)$.

$$\frac{1}{|B_o(n)|}\sum_{y\in B_o(n)} G^n(y,z) \le G^n(o,z).$$

- Green's function $G(x,y)$ and exit time $\mathbb{E}_x[\tau_{B_x(r)}]$ estimates: well-known on $SG$.
- Implement the above into the machinery of Lawler–Bramson–Griffeath '92.

# IDLA shape theorem on $SG$

**Theorem (C.–Huss–Sava-Huss–Teplyaev '17)**

*For every $\epsilon > 0$,*

$$B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon))$$

*holds for all sufficiently large $n$, with probability 1.*

**Inner bound** *proof idea.*

- Define

$$h_n(z) = |B_o(n)| G^n(o,z) - \sum_{y \in B_o(n)} G_n(y,z).$$

  Then $h_n$ solves the Dirichlet problem

$$\begin{cases} \Delta h_n = 1 - |B_o(n)| \mathbf{1}_o, & \text{on } B_o(n), \\ h_n = 0, & \text{on } (B_o(n))^c. \end{cases}$$

- Green's function $G(x,y)$ and exit time $\mathbb{E}_x[\tau_{B_x(r)}]$ estimates: well-known on $SG$.
- Implement the above into the machinery of Lawler–Bramson–Griffeath '92.

# IDLA shape theorem on $SG$

**Theorem (C.–Huss–Sava-Huss–Teplyaev '17)**

*For every* $\epsilon > 0$,

$$B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon))$$

*holds for all sufficiently large n, with probability* 1.

**Inner bound** *proof idea.*

- Define

$$h_n(z) = |B_o(n)|G^n(o,z) - \sum_{y \in B_o(n)} G_n(y,z).$$

  **Divisible sandpile odometer problem**

$$\begin{cases} \Delta u_n = 1 - |B_o(n)|\mathbf{1}_o, & \text{on } S(n), \\ u_n = 0, & \text{on } (S(n))^c. \end{cases}$$

- Green's function $G(x,y)$ and exit time $\mathbb{E}_x[\tau_{B_x(r)}]$ estimates: well-known on $SG$.
- Implement the above into the machinery of Lawler–Bramson–Griffeath '92.

# IDLA shape theorem on $SG$

**Theorem (C.–Huss–Sava-Huss–Teplyaev '17)**

*For every* $\epsilon > 0$,

$$B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon))$$

*holds for all sufficiently large* $n$, *with probability* 1.

**Outer bound** *proof idea.*

- Adapt the algorithm of Duminil-Copin–Lucas–Yadin–Yehudayoff '13, by pausing the IDLA process when *either* the particle attaches to the aggregate *or* when it exits $B_o(n_j)$, where the $n_j$ is defined inductively
  → using the *abelian* property of the IDLA process.
- With the following inputs, we can then implement the algorithm and use the inner bound to show there are no long outward tentacles, and hence control the outer bound.
  - **Geometric** input: Volume growth of balls and of annuli in $SG$.
  - **Potential theoretic** input: Show that the killed Green's function $G^n(x,y) \geq C(\epsilon) > 0$ for all $x, y \in B_o((1-\epsilon)n)$, thanks to the **elliptic Harnack inequality** (proved by Kigami '01 on $SG$) and a chaining argument.

# Limit shapes for Laplacian growth & sandpiles

**Sierpinski gasket ($SG$)** For all models: Launch from the corner vertex $o$.

| *Model* | *Initial chip #* | *Shape theorem/conjecture* |
|---|---|---|
| IDLA | $\lvert B_o(n)\rvert$ | In/out-radius $n \pm \mathcal{O}(\sqrt{\log n})$ [1,2] |
| RRA | $m$ | In-radius $n_m - 2$, out-radius $n_m$ [2] |
| DSM | $m$ | In-radius $n_m - 1$, out-radius $n_m$ [3] |
| ASM | $m$ | ??? |

[1] C.–Huss–Sava-Huss–Teplyaev '17

[2] C.–Kudler-Flam '18

[3] Huss–Sava-Huss '17

# Limit shapes for Laplacian growth & sandpiles

**Sierpinski gasket ($SG$)** For all models: Launch from the corner vertex $o$.

| *Model* | *Initial chip #* | *Shape theorem/conjecture* |
|---|---|---|
| IDLA | $\lvert B_o(n)\rvert$ | In/out-radius $n \pm \mathcal{O}(\sqrt{\log n})$ [1,2] |
| RRA | $m$ | In-radius $n_m - 2$, out-radius $n_m$ [2] |
| DSM | $m$ | In-radius $n_m - 1$, out-radius $n_m$ [3] |
| ASM | $m$ | Receiving set is a ball $B_o(r_m)$<br>$r_m = m^{1/d_H}[\mathcal{G}(\log m) + o(1)]$ as $m \to \infty$ [2]<br>($\mathcal{G}$ is an explicit $(\log 3)$-periodic function) |

---

[1] C.–Huss–Sava-Huss–Teplyaev '17

[2] C.–Kudler-Flam '18

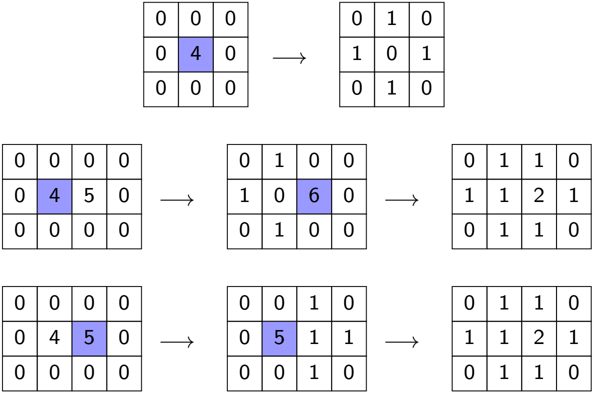
[3] Huss–Sava-Huss '17

**Theorem (Limit shape universality on $SG$)**

*On $SG$, clusters in all 4 single-source growth models fill balls in the graph metric.*

**#1 nontrivial non-tree state space** where **limit shape universality** has been proven.

# The abelian sandpile model [Bak–Tang–Wiesenfeld '87, Dhar, Majumdar, ...]



| | | |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 4 | 0 |
| 0 | 0 | 0 |

⟶

| | | |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 0 |

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 4 | 5 | 0 |
| 0 | 0 | 0 | 0 |

⟶

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 6 | 0 |
| 0 | 1 | 0 | 0 |

⟶

| | | | |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 1 | 1 | 2 | 1 |
| 0 | 1 | 1 | 0 |

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 4 | 5 | 0 |
| 0 | 0 | 0 | 0 |

⟶

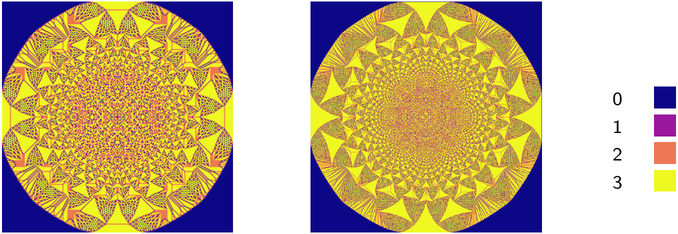
| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 0 | 5 | 1 | 1 |
| 0 | 0 | 1 | 0 |

⟶

| | | | |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 1 | 1 | 2 | 1 |
| 0 | 1 | 1 | 0 |

# Sandpile growth on $\mathbb{Z}^2$: **Fractals in a sandpile**

Lay $m$ chips at the origin and stabilize. Rescale the lattice/cluster by $m^{1/d}$ in length.

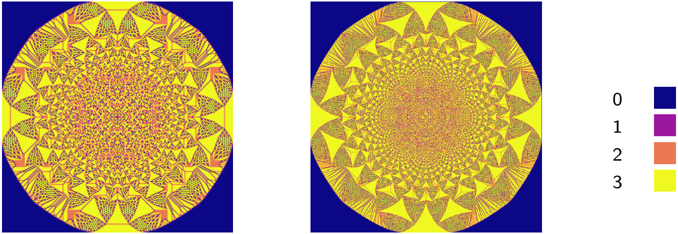


$m = 10^5$ $\qquad$ $m = 10^6$

- Scaling limit of the patterns in weak-$*$ $L_\infty(\mathbb{R}^d)$. [Pegden–Smart '11]
- Apollonian gaskets in the pattern. [Numerically observed since 90s, proved by Levine–Pegden–Smart '12, '14. Latter is published in *Ann. Math.* '17]
  $\rightarrow$ Odometer function satisfies a "sandpile PDE" (integer superharmonic matrices).
- Stability of patterns [Pegden–Smart '17]
- The limit shape is NOT an Euclidean sphere, but rather close to a dodecagon. [No proofs yet]

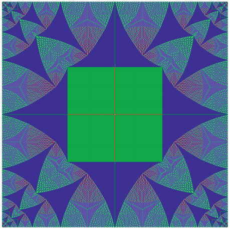
# Sandpile growth on $\mathbb{Z}^2$: **Fractals in a sandpile**

Lay $m$ chips at the origin and stabilize. Rescale the lattice/cluster by $m^{1/d}$ in length.

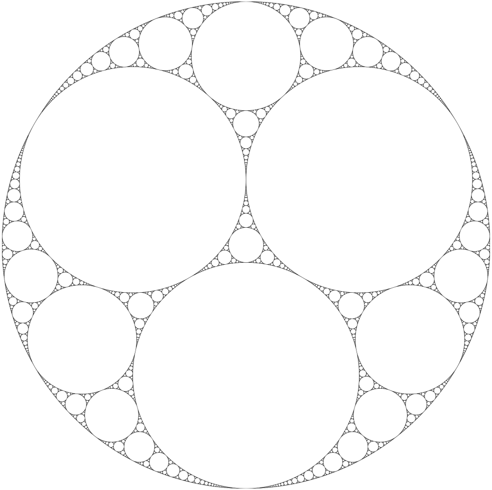


$m = 10^5$ $m = 10^6$

- Identity element of the sandpile group on a sinked finite square lattice.

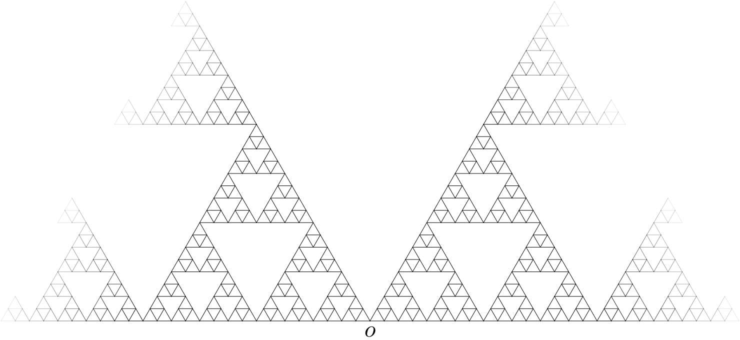


# Apollonian circle packings / gasket



By Time3000 [GFDL or CC BY-SA 4.0-3.0-2.5-2.0-1.0], from Wikimedia Commons
`https://commons.wikimedia.org/wiki/File:Apollonian_gasket.svg`

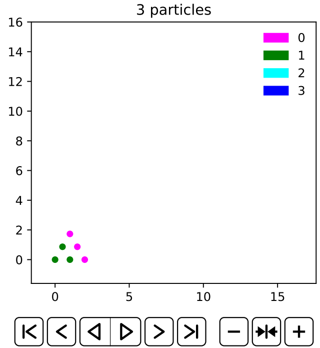
# Let's play sandpiles on (the non-curved) *SG*!



**Q.** Is it possible to see fractal patterns in the sandpile on a fractal?

- It does not matter whether to solve it on the 1-sided *SG* or the 2-sided *SG* (symmetry).

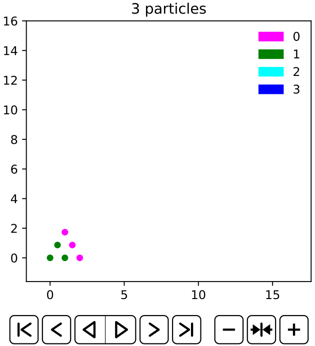
# Let’s play sandpiles on (the non-curved) *SG*!



**Key observations from simulations:**

- Sandpile patterns exhibit periodicity.
- The set of all vertices receiving at least 1 chip is ALWAYS a ball in the graph metric.
- Radial explosions occur at periodic values of $m$. (Not seen on $\mathbb{Z}^d$ or trees!)

# Let's play sandpiles on (the non-curved) *SG*!



**Key observations from simulations:**

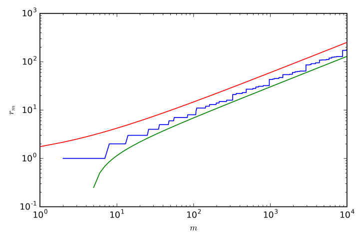
- Sandpile patterns exhibit periodicity.
- The set of all vertices receiving at least 1 chip is ALWAYS a ball in the graph metric.
- Radial explosions occur at periodic values of $m$. (Not seen on $\mathbb{Z}^d$ or trees!)

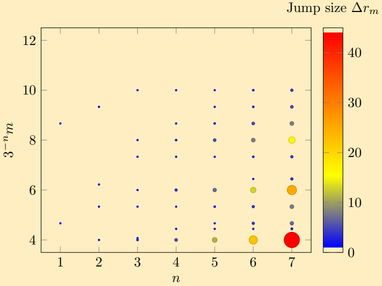
**Numerical work turned into insights & theorems!**

- [Fairchild–Haim–Setra–Strichartz–Westura, arXiv:1602.03424] identified the sandpile growth mechanism.
- [C.–Kudler-Flam, arXiv:1807.08748] solved the sandpile growth problem EXACTLY.

# Top-level abelian sandpile results: radial growth & explosion

Amazing (!!!) numerical discovery by Jonah Kudler-Flam (as a Colgate senior, May '17)



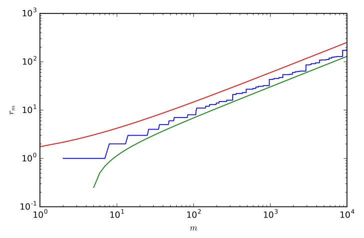


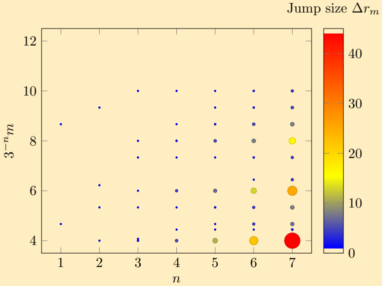
- Major jumps at $4 \cdot 3^n$, $6 \cdot 3^n$, $8 \cdot 3^n$, $10 \cdot 3^n$. Period $= 2 \cdot 3^n$.
- Radial jumps occur at well-defined values of *m* (do not get denser as *n* increases).

# Top-level abelian sandpile results: radial growth & explosion

As a responsible mathematician, the best thing I can do is to prove...

## Theorem (C.–Kudler-Flam '18)

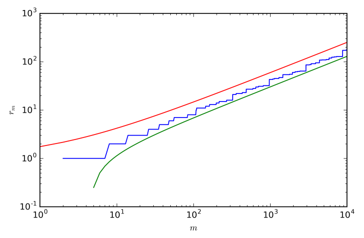


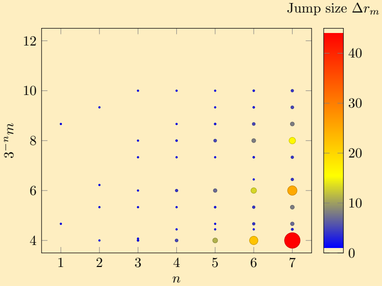


- Major jumps at $4 \cdot 3^n,\ 6 \cdot 3^n,\ 8 \cdot 3^n,\ 10 \cdot 3^n$. Period $= 2 \cdot 3^n$.
- Radial jumps occur at well-defined values of $m$ (do not get denser as $n$ increases).

# Top-level abelian sandpile results: radial scaling limit

## Theorem (C.–Kudler-Flam '18)





An important observation from the numerics is

**Remainder Lemma.** $R(m) := r(3m) - 2r(m) \in \{-1, 0, +1\}$ for all $m$.

# Top-level abelian sandpile results: radial scaling limit

**Remainder Lemma.** $R(m) := r(3m) - 2r(m) \in \{-1, 0, +1\}$ for all $m$.

Assume the Remainder Lemma holds. Recall

## Theorem (Renewal theorem [*cf.* Feller; Falconer, Techniques in fractal geometry])

*Let* $g : \mathbb{R} \to \mathbb{R}$, *and* $\mu$ *be a Borel probability measure supported on* $[0, \infty)$, *Suppose:*

1. $\lambda := \int_0^\infty t \, d\mu(t) < \infty$.
2. $\int_0^\infty e^{-at} \, d\mu(t) < 1$ *for every* $a > 0$.
3. $g$ *has a discrete set of discontinuities, and there exist* $c, \alpha > 0$ *such that* $|g(t)| \le ce^{-\alpha|t|}$ *for all* $t \in \mathbb{R}$.

*Then there is a unique* $f \in \mathcal{F}$ *which solves the* **renewal equation**

$$f(t) = g(t) + \int_0^\infty f(t-y) \, d\mu(y) \qquad (t \in \mathbb{R})$$

*and the solution is*

$$f(t) = \sum_{k=0}^\infty \left(g * \mu^{*k}\right)(t).$$

# Top-level abelian sandpile results: radial scaling limit

**Remainder Lemma.** $R(m) := r(3m) - 2r(m) \in \{-1, 0, +1\}$ for all $m$.

## Theorem (Renewal theorem [*cf.* Feller; Falconer, Techniques in fractal geometry])

**Renewal equation**

$$f(t) = g(t) + \int_0^\infty f(t-y)\, d\mu(y) \qquad (t \in \mathbb{R})$$

*with solution*

$$f(t) = \sum_{k=0}^{\infty} \left(g * \mu^{*k}\right)(t).$$

*$\mu$ is said to be $\tau$-**arithmetic** if $\tau > 0$ is the largest number such that $\mathrm{supp}(\mu) \subset \tau\mathbb{Z}$. If no such $\tau$ exists, $\mu$ is said to be **non-arithmetic**.*

- *If $\mu$ is $\tau$-**arithmetic**, then for all $y \in [0, \tau)$,*

$$\lim_{k\to\infty} f(k\tau + y) = \frac{\tau}{\lambda} \sum_{j=-\infty}^{\infty} g(j\tau + y).$$

- *If $\mu$ is **non-arithmetic**, then*

$$\lim_{t\to\infty} f(t) = \frac{1}{\lambda} \int_{-\infty}^{\infty} g(y)\, dy.$$

# Top-level abelian sandpile results: radial scaling limit

**Remainder Lemma.** $R(m) := r(3m) - 2r(m) \in \{-1, 0, +1\}$ for all $m$.

### Theorem (Renewal theorem [*cf.* Feller; Falconer, Techniques in fractal geometry])

- *If* $\mu$ *is* $\tau$-**arithmetic**, *then for all* $y \in [0, \tau)$,

$$\lim_{k\to\infty} f(k\tau + y) = \frac{\tau}{\lambda} \sum_{j=-\infty}^{\infty} g(j\tau + y).$$

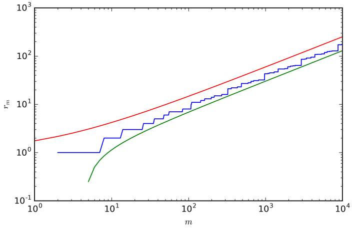
For us, $f(t) = e^{-t/d_H} r(e^t)$, $g(t) = e^{-t/d_H} R(e^t)$, $\mu = \delta_{\log 3}$.

### Theorem (Radial scaling limit)

$$r(x) = x^{1/d_H}[\mathcal{G}(\log x) + o(1)] \quad \textit{as } x \to \infty,$$

*where* $d_H = \log_2 3$ *is the Hausdorff dimension of SG, and* $\mathcal{G}$ *is a* $\log 3$*-periodic function.*

# Top-level abelian sandpile results: radial scaling limit



## Theorem (Radial scaling limit)

$$r(x) = x^{1/d_H}[\mathcal{G}(\log x) + o(1)] \quad \textit{as } x \to \infty,$$

*where* $d_H = \log_2 3$ *is the Hausdorff dimension of SG, and* $\mathcal{G}$ *is an explicit non-constant* $\log 3$*-periodic function. (This uses a separate argument.)*

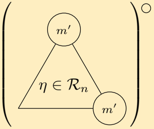
Best possible limit theorem on a state space with discrete scale invariance

**To come:** The radial oscillations are connected to changes in the sandpile patterns.

# The fundamental sandpile diagram on $SG$

**Proposition.** For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ =$$



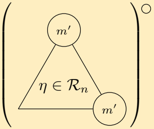
$$\subseteq G_{n+1}.$$

From this follows the **radial equation** $r_m = 2^n + r_{m'-2}$.

# The fundamental sandpile diagram on $SG$

**Proposition.** For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ =$$



$$\subseteq G_{n+1}.$$

From this follows the **radial equation** $r_m = 2^n + r_{m'-2}$.

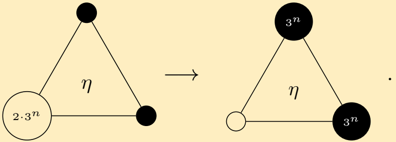
*Key idea:* **systematic topplings in waves**

- $\mathcal{R}_n$ is the **sandpile group** of $G_n$ (= class of **recurrent** sandpile configs on $G_n$) with two sink vertices (cut points) $\partial G_n$. It is an abelian group under $\oplus$, addition of chips followed by stabilization.
- Stabilize $m\mathbb{1}_o$ in $G_n \setminus \partial G_n$ to obtain the config $\eta \in \mathcal{R}_n$, pausing the excess $m'$ chips on $\partial G_n$ (sink) → using the abelian property.
- Then topple the excess chips on $\partial G_n$ (source). By Dhar's multiplication by identity test (a Laplacian identity), with each topple on $\partial G_n$, $\eta$ is unchanged, while the # of chips on $\partial G_n$ decrements in steps of 2, until 2 or 3 chips remain.

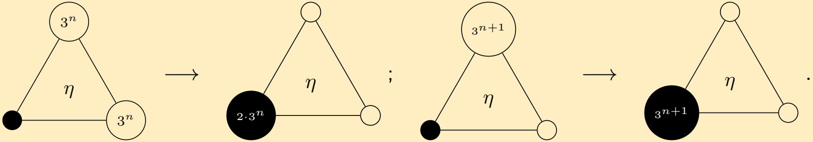
Using this Proposition, we can inductively prove: cluster shape is a ball; periodicity; etc.

# Toppling identities, periodicity

**Proposition.** For every $\eta \in \mathcal{R}_n^{(s)}$,



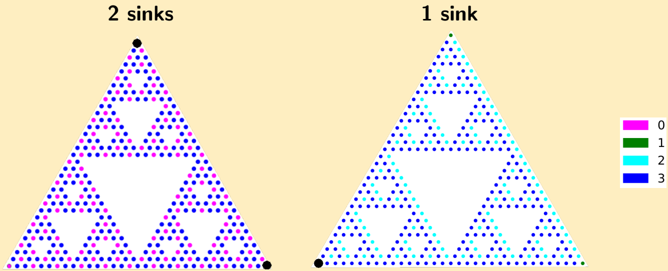
For every $\eta \in \mathcal{R}_n^{(o)}$,



This explains the $(2 \cdot 3^n)$-periodicity in the sandpile growth (and the patterns restricted to $G_n$).

# Identity element of the sandpile group of *SG*

## Theorem (Identity elements)

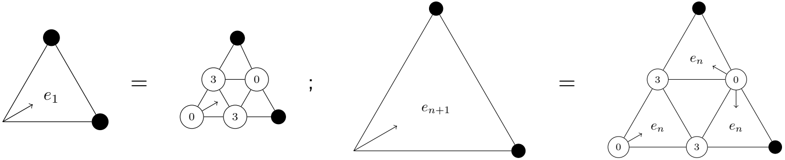


**2 sinks**

**1 sink**

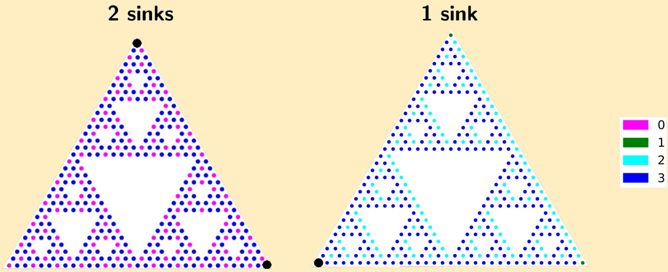
$e_5$

$e_5^{(o)} := M_5 \oplus (\mathbf{1}_x + \mathbf{1}_y)$



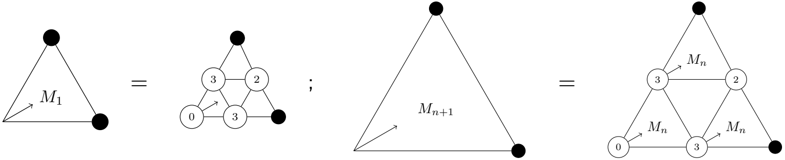
# Identity element of the sandpile group of *SG*

## Theorem (Identity elements)



$e_5$

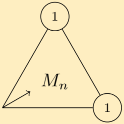
$e_5^{(o)} := M_5 \oplus (\mathbf{1}_x + \mathbf{1}_y)$



# The main explosions at $4 \cdot 3^n$: transition from $M_n$ to $e_n$

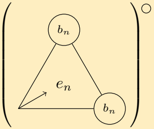
**Proposition.**

$$((4 \cdot 3^n - 2)\mathbb{1}_o)^\circ =$$



;

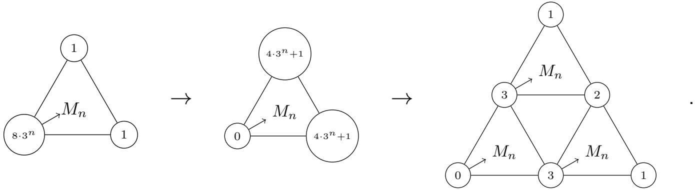
$$((4 \cdot 3^n)\mathbb{1}_o)^\circ =$$



, where $b_n = \frac{3}{2}\left(3^{n-1}+1\right)$.

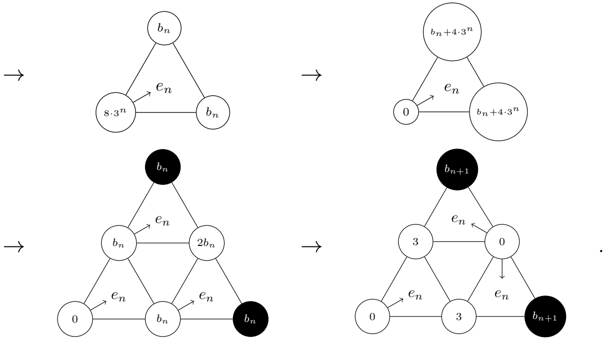
*Proof of the induction step.*

$(4 \cdot 3^{n+1} - 2)\mathbb{1}_o \to$

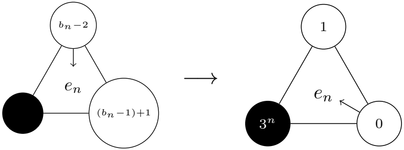


*Proof (cont.).*

$(4 \cdot 3^{n+1})\mathbb{1}_o \rightarrow$



The last step depends on the following "reflection lemma," using the axial symmetry of *SG*:



Also note $b_{n+1} = b_n + 3^n$.

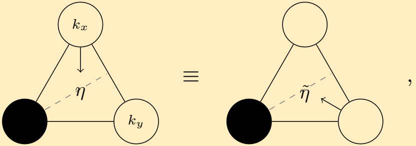
# An axial reflection lemma (*SG* with one sink)

$e_n^{(o)}$ is the id element of $\mathcal{R}_n^{(o)}$; $\partial G_n = \{x, y\}$.

**Lemma.** Let $\eta \in \mathcal{R}_n^{(o)}$ be such that $\eta = e_n^{(o)} \oplus \alpha \mathbb{1}_x \oplus \beta \mathbb{1}_y$ for some $\alpha, \beta \in \mathbb{N}_0$. Let $k_x, k_y \in \mathbb{N}_o$ solve the system of equations

$$\begin{cases} \alpha + k_x & = \beta + p_0 \cdot 3^n + p_1 \cdot 3^{n+1} \\ \beta + k_y & = \alpha + p_0 \cdot 3^n + p_2 \cdot 3^{n+1} \end{cases}$$

for some $p_0, p_1, p_2 \in \mathbb{Z}$ (which come from the toppling identities).
Then



where $\tilde{\eta} = e_n^{(o)} \oplus \beta \mathbb{1}_x \oplus \alpha \mathbb{1}_y$ is the reflection of $\eta$ across the axis of symmetry.

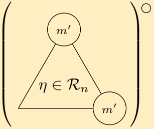
For the example on the previous slide:

$$\alpha = 2 \cdot 3^n, \ \beta = b_n - 2, \ k_x = b_n - 2, \ k_y = b_n - 1, \ p_0 = -1, \ p_1 = 1, \ p_2 = 0.$$

# The fundamental sandpile diagram on $SG$

**Proposition.** For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ =$$



$$\subseteq G_{n+1}.$$

From this follows the fundamental equation $r_m = 2^n + r_{m'-2}$.

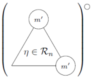
*Example:* $n = 3$. Record values of $m$ at which $m'$ jumps.

| $\frac{m}{3^n}$ | $m$ | $m'$ | $m-2m'$ | $\Delta r_m$ |
|---|---|---|---|---|
| **4** | **108** | **15** | **78** | **2** |
| $4\frac{2}{27}$ | 110 | 16 | 78 | 1 |
| $4\frac{4}{9}$ | 120 | 19 | 82 | |
| $4\frac{2}{3}$ | 126 | 20 | 86 | |
| $5\frac{1}{3}$ | 144 | 28 | 88 | 1 |
| 6 | 162 | 42 | 78 | 1 |
| $6\frac{2}{27}$ | 164 | 43 | 78 | |
| $6\frac{4}{9}$ | 174 | 46 | 82 | |
| $6\frac{2}{3}$ | 180 | 47 | 86 | |
| $7\frac{1}{3}$ | 198 | 55 | 88 | 1 |

| $\frac{m}{3^n}$ | $m$ | $m'$ | $m-2m'$ | $\Delta r_m$ |
|---|---|---|---|---|
| 8 | 216 | 69 | 78 | 1 |
| $8\frac{2}{27}$ | 218 | 70 | 78 | |
| $8\frac{4}{9}$ | 228 | 73 | 82 | |
| $8\frac{2}{3}$ | 234 | 74 | 86 | |
| $9\frac{1}{3}$ | 252 | 82 | 88 | |
| 10 | 270 | 96 | 78 | 1 |
| $10\frac{2}{27}$ | 272 | 97 | 78 | |
| $10\frac{4}{9}$ | 282 | 100 | 82 | |
| $10\frac{2}{3}$ | 288 | 101 | 86 | |
| $11\frac{1}{3}$ | 306 | 109 | 88 | |

| $\frac{m}{3^n}$ | $m$ | $m'$ | $m-2m'$ | $\Delta r_m$ |
|---|---|---|---|---|
|  | 2 | 1 | 0 | 1 |
|  | 8 | 4 | 0 | 1 |
|  |  | **n = 1** |  |  |
| **4** | **12** | **3** | **6** |  |
| $4\frac{2}{3}$ | 14 | 4 | 6 | 1 |
| 6 | 18 | 6 | 6 |  |
| $6\frac{2}{3}$ | 20 | 7 | 6 |  |
| 8 | 24 | 9 | 6 |  |
| $8\frac{2}{3}$ | 26 | 10 | 6 | 1 |
| 10 | 30 | 12 | 6 |  |
| $10\frac{2}{3}$ | 32 | 13 | 6 |  |
|  |  | **n = 2** |  |  |
| **4** | **36** | **6** | **24** | **1** |
| $4\frac{2}{9}$ | 38 | 7 | 24 |  |
| $4\frac{2}{3}$ | 42 | 8 | 26 |  |
| $5\frac{1}{3}$ | 48 | 10 | 28 | 1 |
| 6 | 54 | 15 | 24 |  |
| $6\frac{2}{9}$ | 56 | 16 | 24 | 1 |
| $6\frac{2}{3}$ | 60 | 17 | 26 |  |
| $7\frac{1}{3}$ | 66 | 19 | 28 |  |
| 8 | 72 | 24 | 24 |  |
| $8\frac{2}{9}$ | 74 | 25 | 24 |  |
| $8\frac{2}{3}$ | 78 | 26 | 26 |  |
| $9\frac{1}{3}$ | 84 | 28 | 28 | 1 |
| 10 | 90 | 33 | 24 |  |
| $10\frac{2}{9}$ | 92 | 34 | 24 |  |
| $10\frac{2}{3}$ | 96 | 35 | 26 |  |
| $11\frac{1}{3}$ | 102 | 37 | 28 |  |
|  |  | **n = 3** |  |  |
| **4** | **108** | **15** | **78** | **2** |
| $4\frac{2}{27}$ | 110 | 16 | 78 | 1 |
| $4\frac{4}{9}$ | 120 | 19 | 82 |  |
| $4\frac{2}{3}$ | 126 | 20 | 86 |  |
| $5\frac{1}{3}$ | 144 | 28 | 88 | 1 |
| 6 | 162 | 42 | 78 | 1 |
| $6\frac{2}{27}$ | 164 | 43 | 78 |  |
| $6\frac{4}{9}$ | 174 | 46 | 82 |  |
| $6\frac{2}{3}$ | 180 | 47 | 86 |  |
| $7\frac{1}{3}$ | 198 | 55 | 88 | 1 |
| 8 | 216 | 69 | 78 | 1 |
| $8\frac{2}{27}$ | 218 | 70 | 78 |  |
| $8\frac{4}{9}$ | 228 | 73 | 82 |  |
| $8\frac{2}{3}$ | 234 | 74 | 86 |  |
| $9\frac{1}{3}$ | 252 | 82 | 88 |  |
| 10 | 270 | 96 | 78 | 1 |
| $10\frac{2}{27}$ | 272 | 97 | 78 |  |
| $10\frac{4}{9}$ | 282 | 100 | 82 |  |
| $10\frac{2}{3}$ | 288 | 101 | 86 |  |
| $11\frac{1}{3}$ | 306 | 109 | 88 |  |
|  |  | **n = 4** |  |  |
| **4** | **324** | **42** | **240** | **5** |
| $4\frac{2}{81}$ | 326 | 43 | 240 |  |
| $4\frac{4}{9}$ | 360 | 55 | 250 | 1 |
| $4\frac{2}{3}$ | 378 | 56 | 266 |  |
| $5\frac{1}{3}$ | 432 | 82 | 268 | 1 |
| 6 | 486 | 123 | 240 | 4 |
| $6\frac{2}{81}$ | 488 | 124 | 240 |  |
| $6\frac{4}{9}$ | 522 | 136 | 250 |  |
| $6\frac{2}{3}$ | 540 | 137 | 266 |  |
| $7\frac{1}{3}$ | 594 | 163 | 268 | 1 |
| 8 | 648 | 204 | 240 | 2 |
| $8\frac{2}{81}$ | 649 | 205 | 240 |  |
| $8\frac{4}{9}$ | 684 | 217 | 250 |  |
| $8\frac{2}{3}$ | 702 | 218 | 266 | 1 |
| $9\frac{1}{3}$ | 756 | 244 | 268 |  |
| 10 | 810 | 285 | 240 | 1 |
| $10\frac{2}{81}$ | 812 | 286 | 240 |  |
| $10\frac{4}{9}$ | 846 | 298 | 250 |  |
| $10\frac{2}{3}$ | 864 | 299 | 266 |  |
| $11\frac{1}{3}$ | 918 | 325 | 268 |  |
|  |  | **n = 5** |  |  |
| **4** | **972** | **123** | **726** | **11** |
| $4\frac{2}{3^5}$ | 974 | 124 | 726 |  |
| $4\frac{4}{9}$ | 1080 | 163 | 754 | 1 |
| $4\frac{2}{3}$ | 1134 | 164 | 806 | 1 |
| $5\frac{1}{3}$ | 1296 | 244 | 808 | 2 |
| 6 | 1458 | 366 | 726 | 7 |
| $6\frac{2}{3^5}$ | 1460 | 367 | 726 |  |
| $6\frac{4}{9}$ | 1566 | 406 | 754 |  |
| $6\frac{2}{3}$ | 1620 | 407 | 806 |  |
| $7\frac{1}{3}$ | 1782 | 487 | 808 | 1 |
| 8 | 1944 | 609 | 726 | 5 |
| $8\frac{2}{3^5}$ | 1946 | 610 | 726 |  |
| $8\frac{4}{9}$ | 2052 | 649 | 754 |  |
| $8\frac{2}{3}$ | 2106 | 650 | 806 | 2 |
| $9\frac{1}{3}$ | 2268 | 730 | 808 | 1 |
| 10 | 2430 | 852 | 726 | 1 |
| $10\frac{2}{3^5}$ | 2432 | 853 | 726 |  |
| $10\frac{4}{9}$ | 2538 | 892 | 754 |  |
| $10\frac{2}{3}$ | 2592 | 893 | 806 |  |
| $11\frac{1}{3}$ | 2754 | 943 | 808 |  |
|  |  | **n = 6** |  |  |
| **4** | **2916** | **366** | **2184** | **22** |
| $4\frac{2}{3^6}$ | 2918 | 367 | 2184 |  |
| $4\frac{4}{9}$ | 3240 | 487 | 2266 | 1 |
| $4\frac{2}{3}$ | 3402 | 488 | 2426 | 4 |
| $5\frac{1}{3}$ | 3888 | 730 | 2428 | 4 |
| 6 | 4374 | 1095 | 2184 | 13 |
| $6\frac{2}{3^6}$ | 4376 | 1096 | 2184 |  |
| $6\frac{4}{9}$ | 4698 | 1216 | 2266 | 1 |
| $6\frac{2}{3}$ | 4860 | 1217 | 2426 |  |
| $7\frac{1}{3}$ | 5346 | 1459 | 2428 | 2 |
| 8 | 5832 | 1824 | 2184 | 8 |
| $8\frac{2}{3^6}$ | 5834 | 1825 | 2184 |  |
| $8\frac{4}{9}$ | 6156 | 1945 | 2266 |  |
| $8\frac{2}{3}$ | 6318 | 1946 | 2426 | 5 |
| $9\frac{1}{3}$ | 6804 | 2188 | 2428 | 2 |
| 10 | 7290 | 2553 | 2184 | 2 |
| $10\frac{2}{3^6}$ | 7292 | 2554 | 2184 |  |
| $10\frac{4}{9}$ | 7614 | 2674 | 2266 |  |
| $10\frac{2}{3}$ | 7766 | 2675 | 2426 |  |
| $11\frac{1}{3}$ | 8262 | 2917 | 2428 |  |
|  |  | **n = 7** |  |  |
| **4** | **8748** | **1095** | **6558** | **44** |
| $4\frac{2}{3^7}$ | 8750 | 1096 | 6558 |  |
| $4\frac{4}{9}$ | 9720 | 1459 | 6802 | 3 |
| $4\frac{2}{3}$ | 10206 | 1460 | 7286 | 7 |
| $5\frac{1}{3}$ | 11664 | 2188 | 7288 | 8 |
| 6 | 13122 | 3282 | 6558 | 25 |

Legend: $(m\mathbb{1}_o)^\circ =$

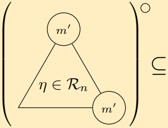


; $\#\{\text{chips in } \eta\} = m - 2m'$.

# Detailed results: radial jumps

## Theorem (Enumeration of radial jumps)

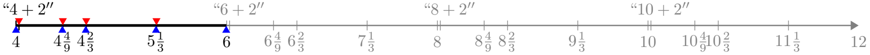
*For* $n \geq 3$ *and* $m \in [4 \cdot 3^n, 4 \cdot 3^{n+1})$, $(m\mathbb{1}_o)^\circ =$



$\subseteq G_{n+1}$, *where* $m \mapsto m'$ *is a piecewise constant right-continuous function which has jumps indicated in the following table.*

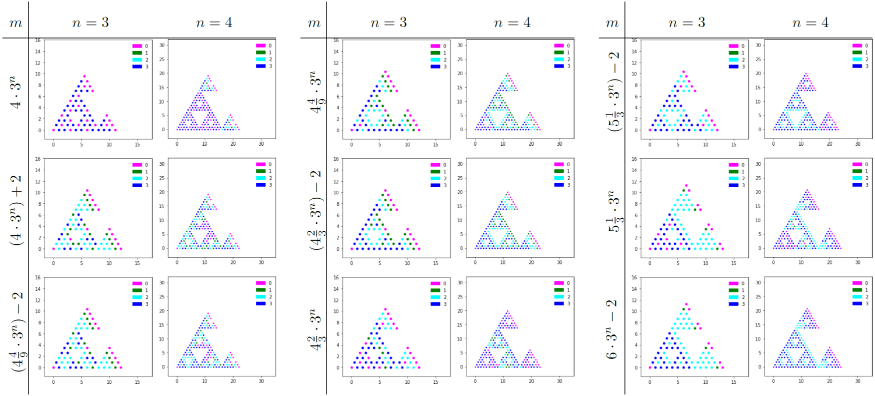
| $m$ | $m'$ |
|---|---|
| $(4 + 2p) \cdot 3^n$ | $b_n + p \cdot 3^n$ |
| $(4 + 2p) \cdot 3^n + 2$ | $(b_n + 1) + p \cdot 3^n$ |
| $(4\frac{4}{9} + 2p) \cdot 3^n$ | $2 \cdot 3^{n-1} + 1 + p \cdot 3^n$ |
| $(4\frac{2}{3} + 2p) \cdot 3^n$ | $2 \cdot 3^{n-1} + 2 + p \cdot 3^n$ |
| $(5\frac{1}{3} + 2p) \cdot 3^n$ | $3^n + 1 + p \cdot 3^n$ |

*where* $p \in \{\mathbf{0}, 1, 2, 3\}$, *and* $b_n = |V(G_{n-1})| = \frac{3}{2}(3^{n-1} + 1)$.



# Detailed results: **Fractals in the sandpile on a fractal**

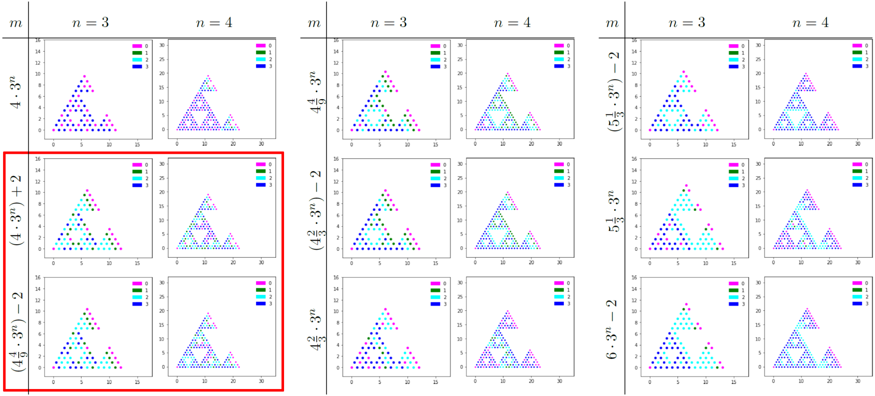
## Patterns associated to the jumps



An inductive proof—**Sandpile block renormalization**
Configuration restricted to $G_n$ is the gluing of 3 well-defined **sandpile tiles**.
[**W**hat **Y**ou **S**ee **I**s **W**hat **Y**ou **G**et]

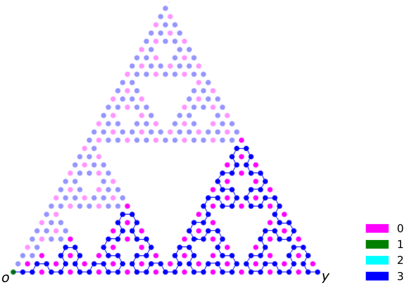
# Detailed results: **Fractals in the sandpile on a fractal**

## Patterns associated to the jumps

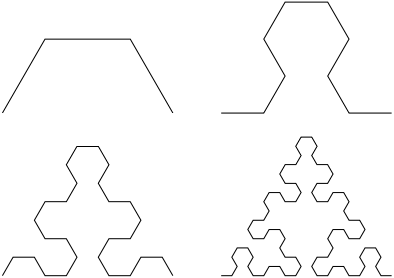


BUT there are two exceptions which cannot be explained by tiling arguments.

# $e_n$: A fractal (Peano curve) within a fractal ($SG$)!

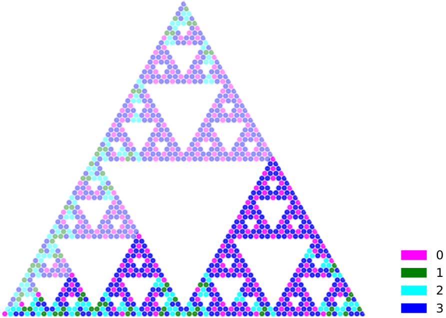


$$e_5 = ((4 \cdot 3^5)\mathbf{1}_o)^\circ$$



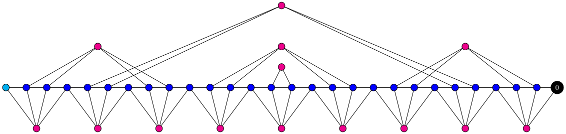
- The unique shortest blue path (a Peano curve formed by the concatenation of the first $n$ Sierpinski arrowhead curves) of 3's connecting $o$ to the sink $y$.
- What happens when 2 chips are added to $o$?
- Triggers a chain reaction of topplings down the Peano curve, all the way to $y$! This sends 1 extra chip to each sink vertex, which explains "$4 \cdot 3^n + 2$" in the radial jump theorem.
- BUT ...

# $e_n + 2$: Traps develop along the Peano curve

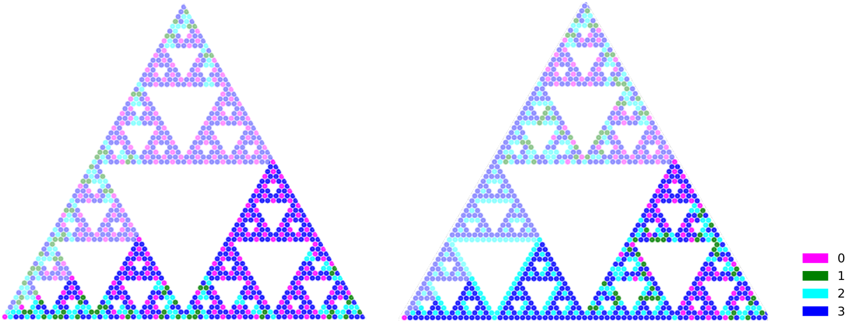


$((4 \cdot 3^6 + 2)\mathbf{1}_o)^\circ$

- Blotches of 1's and 2's ("traps") develop at well-defined locations, due to connections across vertices on the Peano curve.
- Best way to visualize this is to parametrize $SG$ along the length of the Peano curve:



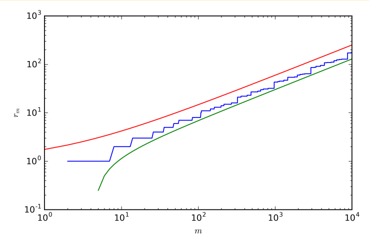
# "$4\frac{4}{9} \cdot 3^n - 2$": Inability to overcome the traps

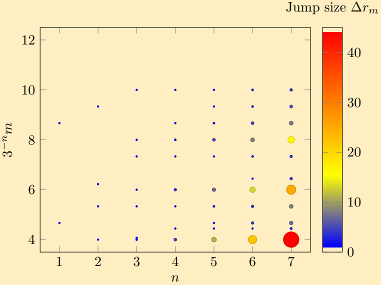


$((4 \cdot 3^6 + 2)\mathbf{1}_o)^\circ$ $((4\frac{4}{9} \cdot 3^6 - 2)\mathbf{1}_o)^\circ$

# Summary: Exact solution of abelian sandpile growth on *SG*

## Theorem (Recursive radial formula)





$$\begin{tikzcd}
 & 6\frac{2}{3} \arrow[d] & & 8\frac{4}{9} \arrow[r] & 7\frac{1}{3} \arrow[d] & 8 \arrow[l] & \\
6\frac{4}{9} \arrow[r] & 4\frac{2}{3} \arrow[r, very thick] & 6 \arrow[r] & 4\frac{4}{9} \arrow[r, very thick] & 5\frac{1}{3} \arrow[r, very thick] & 8\frac{2}{3} \arrow[u] & 10 \arrow[loop right] \\
 & & & 4 \arrow[u, very thick] & & 9\frac{1}{3} \arrow[u] &
\end{tikzcd}$$

$$a \rightarrow b: \quad r_{a\cdot 3^n} = 2^n + r_{b\cdot 3^{n-1}} \quad (n \geq 3)$$
$$a \longrightarrow b: \quad r_{a\cdot 3^n} = 2^n + r_{b\cdot 3^{n-2}} \quad (n \geq 4)$$

The recursive radial formula implies the aforementioned Remainder Lemma.

# Limit shapes for Laplacian growth & sandpiles

**Sierpinski gasket ($SG$)** For all models: Launch from the corner vertex $o$.

| *Model* | *Initial chip #* | *Shape theorem/conjecture* |
|---|---|---|
| IDLA | $\lvert B_o(n)\rvert$ | In/out-radius $n \pm \mathcal{O}(\sqrt{\log n})$ [1,2] |
| RRA | $m$ | In-radius $n_m - 2$, out-radius $n_m$ [2] |
| DSM | $m$ | In-radius $n_m - 1$, out-radius $n_m$ [3] |
| ASM | $m$ | Receiving set is a ball $B_o(r_m)$<br>$r_m = m^{1/d_H}[\mathcal{G}(\log m) + o(1)]$ as $m \to \infty$ [2]<br>($\mathcal{G}$ is an explicit $(\log 3)$-periodic function) |

[1] C.–Huss–Sava-Huss–Teplyaev '17

[2] C.–Kudler-Flam '18

[3] Huss–Sava-Huss '17

### Theorem (Limit shape universality on $SG$)

*On $SG$, clusters in all 4 single-source growth models fill balls in the graph metric.*

**#1 nontrivial non-tree state space** where **limit shape universality** has been proven.

Remark. Ahmed Bou-Rabee has a 2nd example (supercritical percolation cluster on $\mathbb{Z}^2$).

# The mathematical beyond: Open questions

- Fluctuation of IDLA on *SG*: $\mathcal{O}(\sqrt{\log n})$ per simulations
- Log-periodic radial oscillations: beautiful numerics, but proofs?

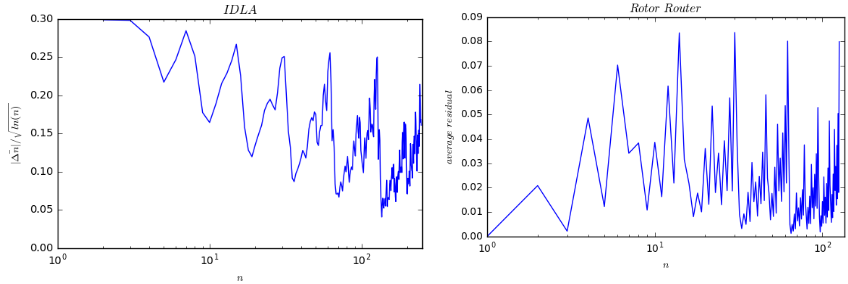


Figure: Sample average of the absolute value of the radial fluctuations about the expected radius.

- Other examples of state spaces on which limit shape universality holds?
  My (naive) conjecture: should hold on any planar nested fractal (as defined by Lindstrom)
- Change the initial condition: Single-source to (random) multi-source?
  *cf.* Abelian sandpile on $\mathbb{Z}^2$ with initial Bernoulli 3-5 configuration (Bou-Rabee)

Thank you!