1 Conformal Property

Definition 1. A Möbius transformation is a function from $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $f(z) = \frac{az+b}{cz+d}$, a, b, c, d $\in \mathbb{C}$, $ad - bc \neq 0$.

Definition 2. f(z) is conformal at z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$.

- The Möbius transformation f(z) has a simple pole at $z = -\frac{d}{c}$ but is analytic everywhere else.
- $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0 \Rightarrow$ f(z) is conformal everywhere except at its pole.

2 Matrix Representations

We can associate any invertible 2 x 2 matrix with a Möbius transformation under the mapping

$$\pi : GL(2, \mathbb{C}) \to MG$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) := \frac{az+b}{cz+d}$$

where $GL(2, \mathbb{C})$ is the group of invertible 2 x 2 matrices and MG is the group of Möbius transformations. Note that any matrices related by scalar multiples map to the same Möbius transformation.

Definition 3. The map Ψ is a group homomorphism from G_1 to G_2 if $\Psi(g)\Psi(h) = \Psi(gh)$ $\forall\ g, h \in G_1$.

Definition 4. The map Ψ is a group isomorphism if it is both a homomorphism and a bijection. Two groups are isomorphic if there exists an isomorphism from one group to the other (denoted $\cong$).

- The map π is therefore a homomorphism but not an isomorphism from $GL(2, \mathbb{C})$ to MG because it is not injective.

Definition 5. Let G be a group and H a normal subgroup of G. Then G/H = {aH: a $\in$ G}.

Definition 6. $PGL(2, \mathbb{C}) \equiv GL(2, \mathbb{C})/Z(GL(2, \mathbb{C}))$ (Projective linear group), $PSL(2, \mathbb{C}) \equiv SL(2, \mathbb{C})/Z(SL(2, \mathbb{C}))$ (Projective special linear group).

Theorem 1. Let $\Psi : G_1 \to G_2$ be a surjective homomorphism. Then $G_1/ker(\Psi) \cong G_2$.

- ker(π) = k$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, k $\in \mathbb{C} \setminus \{0\}$.
- ker(π) = Z($GL(2, \mathbb{C})$). $\Rightarrow PGL(2, \mathbb{C}) \cong MG$.

Now associate any invertible 2 x 2 matrix with determinant 1 to with a Möbius transformation under the homomorphism

$$\phi : SL(2,\mathbb{C}) \to MG$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) := \frac{az+b}{cz+d}$$

- $\ker(\phi) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also, $\mathrm{Z}(GL(2,\mathbb{C})) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- By the homomorphism theorem, MG $\cong$ SL(2,$\mathbb{C}$)/ker(ϕ). $\Rightarrow MG \cong PSL(2,\mathbb{C})$.
- Because $\ker(\phi) \cong \mathbb{Z}_2$, we find that $SL(2,\mathbb{C})$ is just a double covering of the Möbius Group.
- Likewise $GL(2,\mathbb{C})$, which we have been using to represent the Möbius Group, actually covers the Möbius Group an uncountably infinite number of times. This makes sense because $GL(2,\mathbb{C})$ has 4 continuous complex degrees of freedom while $SL(2,\mathbb{C})$ and the Möbius Group only have 3.